Amicable quadruples.

$7\cdot107\cdot a$, $11\cdot71\cdot a$, $17\cdot47\cdot a$, $863\cdot a$ $(a = 2^8\cdot3^3\cdot5\cdot37\cdot73)$
$17\cdot79\cdot b$, $19\cdot71\cdot b$, $23\cdot59\cdot b$, $1439\cdot b$ $(b = 2^5\cdot3^5\cdot5\cdot13)$
$19\cdot71\cdot c$, $23\cdot59\cdot c$, $29\cdot47\cdot c$, $1439\cdot c$ $(c = 2^{15}\cdot3\cdot5^2\cdot11\cdot31\cdot43\cdot257)$
$5\cdot139\cdot d$, $13\cdot59\cdot d$, $19\cdot41\cdot d$, $839\cdot d$, $(d = 2^{33}\cdot3^6\cdot7\cdot23\cdot83\cdot137\cdot331\cdot547\cdot1093\cdot43691\cdot131071)$,
$(d = 2^{33}\cdot3^{10}\cdot7\cdot23\cdot83\cdot107\cdot331\cdot3851\cdot43691\cdot131071)$,
$(d = 2^{37}\cdot3^6\cdot7\cdot23\cdot83\cdot137\cdot331\cdot547\cdot1093\cdot43691\cdot174763\cdot524287)$ and
$(d = 2^{37}\cdot3^{10}\cdot7\cdot23\cdot83\cdot107\cdot331\cdot3851\cdot43691\cdot174763\cdot524287)$.
$17\cdot1999\cdot e$, $23\cdot1499\cdot e$, $71\cdot499\cdot e$, $149\cdot239\cdot e$, $(e = 2^9\cdot3^2\cdot7\cdot11\cdot13)$, $(e=2^9\cdot3^3\cdot5\cdot11)$,
$(e = 2^{14}\cdot3\cdot5\cdot7\cdot19\cdot151)$, $(e = 2^{14}\cdot3^2\cdot7^2\cdot13\cdot19^2\cdot127\cdot151)$,
$(e = 2^{25}\cdot3^3\cdot5^2\cdot19\cdot683\cdot2731\cdot8191)$, $(e = 2^{33}\cdot3^4\cdot7\cdot11^3\cdot61\cdot83\cdot331\cdot43691\cdot131071)$
and $(e = 2^{37}\cdot3^4\cdot7\cdot11^3\cdot61\cdot83\cdot331\cdot43691\cdot174763\cdot524287)$.

Amicable quintuples.

$17\cdot79\cdot a$, $19\cdot71\cdot a$, $23\cdot59\cdot a$, $29\cdot47\cdot a$, $1439\cdot a$ $(a = 2^{15}\cdot3^5\cdot5^2\cdot7\cdot11\cdot43\cdot257)$.
$17\cdot439\cdot b$, $29\cdot263\cdot b$, $43\cdot179\cdot b$, $59\cdot131\cdot b$, $71\cdot109\cdot b$,
$(b = 2^{33}\cdot3^6\cdot5\cdot7\cdot11^2\cdot19\cdot83\cdot331\cdot43691\cdot13107)$ and
$(b = 2^{37}\cdot3^6\cdot5\cdot7\cdot11^2\cdot19\cdot83\cdot331\cdot43691\cdot174763\cdot524287)$.

Amicable sextuples.

$17\cdot1999\cdot a$, $23\cdot1499\cdot a$, $59\cdot599\cdot a$, $71\cdot499\cdot a$, $79\cdot449\cdot a$, $35999\cdot a$
$(a = 2^{36}\cdot3^9\cdot5^2\cdot7^5\cdot11^2\cdot13\cdot19^2\cdot31^2\cdot61\cdot83\cdot127\cdot223\cdot331\cdot7019\cdot112303\cdot898423\cdot616318177)$.

$23\cdot1499\cdot b$, $59\cdot599\cdot b$, $71\cdot499\cdot b$, $79\cdot449\cdot b$, $149\cdot239\cdot b$, $179\cdot199\cdot b$, where b is to be chosen as follows: take any multiply perfect number of multiplicity 6 which contains the factors $79\cdot157$ and which does not contain any of the other factors multiplying b in the numbers above, from this number omit $79\cdot157$ and what remains will give a value for b. The known multiply perfect numbers furnish at least six such values of b.

2. **Multiply Amicable Number Sets.** If we seek for multiply amicable number pairs of the form apq and arf, where p, q and r are primes and a and f are composite, but prime to each other and to p, q and r, we have the equations

$$S(a)S(pq) = S(a)S(rf) = ta(pq + rf).$$

By the same process as that used in § 1 we arrive at equation (E), using the same substitutions, except that we replace $ta/(2ta - S(a))$ by b/c. The numbers p, q, and r depend only on the ratio b/c. It is evident, therefore, that the same numbers p, q, r and f that will form an amicable number pair with a given a will serve to form a multiply amicable number pair if we can find a new a' such that $ta'/(2ta' - S(a')) = a/(2a - S(a)) = b/c$. For example, the numbers $a\cdot11\cdot41\cdot173$

and $a \cdot 71 \cdot 1217$ form

an amicable number pair for ..... $a = 2^3$,
a multiply amicable pair ($t = 2$) for ..... $a = 2^{10} \cdot 3^2 \cdot 7 \cdot 13 \cdot 23 \cdot 89$,
a multiply amicable pair ($t = 3$) for
$a = 2^{21} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 13^3 \cdot 17 \cdot 19^3 \cdot 23 \cdot 89 \cdot 181 \cdot 379 \cdot 683 \cdot 757$.

A table of multiply perfect numbers can be used in connection with formula (E) to find multiply amicable numbers. In case $f = 1$, formula (E) reduces to[1] $(cx - b)(cy - b) = b^2$. An illustration with this simpler formula will be given. Take any multiply perfect number of multiplicity 4 containing the factor 31, for example, $2^9 \cdot 3^3 \cdot 5 \cdot 11 \cdot 31$, and choose for a all of this except the factor 31, $a = 2^9 \cdot 3^3 \cdot 5 \cdot 11$. For $t = 2$ this will give $b = 16$, $c = 1$. Using these values of b and c we get $p = 23$, $q = 47$, $r = 1151$. This gives the multiply amicable pair $23 \cdot 47 \cdot a$ and $1151 \cdot a$ of multiplicity 2. It is necessary that the number a do not already contain any of the factors p, q and r. Since the values of b and c depend only on the fact that the multiply perfect number is of multiplicity 4 and contains the factor 31, any number having these properties will give a value for a. It can readily be seen that if we had taken a multiply perfect number of multiplicity 6 in the above illustration and had used $t = 3$ we should have found $b = 16, c = 1$. This would have given the same values $p = 23, q = 47, r = 1151$, and hence, apq and ar would be a multiply amicable pair of multiplicity 3.

If we omit from any multiply perfect number of multiplicity $2t$ a factor found in column numbered 1, in the table below, and use the remainder of the number for a value of a, then a taken with the corresponding set in column 2 will give a multiply amicable number pair of multiplicity t, provided that a does not contain the factors in column 2.

1	2		1	2	
5	3·11	47	31	17·137·2990783	10103·735263
7	5·11	71	31	17·167·13679	809·51071
11	7·23	191	31	17·137·262079	12959·50231
19	11·59	719	59	31·479	15359
23	19·29	599	71	59·89	5399
29	17·89	1619	79	41·839	35279
31	19·1439	149·191	79	47·239	11519
31	23·47	1151	89	59·179	10799
31	23·47·9767	1583·7103	103	59·389	23399
31	47·89	53·79	139	83·419	35279
31	23·467	103·107	149	89·449	40499
31	23·479	89·127	191	97·4703	460991
31	23·1367	53·607	307	167·1847	310463
31	17·5119	239·383	463	239·6959	1670399
31	17·10303	167·1103			

[1] This is not a particular case of equation (E) but is obtained in the same way, putting $p + 1 = x$ and $q + 1 = y$; that is, setting $g = h = 1$, as well as f.—EDITOR.

By use of this table in connection with a table[1] of the known multiply perfect numbers about six hundred multiply amicable number pairs can be obtained of multiplicity $t = 2, 3$.

Amicable number sets of the form am and an can be found by the use of the equations $S(m) = S(n)$ and $S(a)S(m) = ta(m + n)$. From the latter equation we have $S(a)/ta = (m + n)/S(m)$. Select m and n so that $S(m) = S(n)$ and then find the ratio $(m + n)/S(m)$ and seek a value for a. This method will serve for amicable k-tuples.

For the case of multiplicity 2 we have the following[2]: If m and n are amicable numbers and a is a perfect number, prime to m and n, then am and an are multiply amicable numbers of multiplicity 2. By hypothesis $S(a) = 2a$ and $S(m) = S(n) = m + n$, and hence, $S(a)S(m) = S(a)S(n) = 2a(m + n)$.

A few multiply amicable number sets are given here that were not found by means of the table above.

For $t = 2$:

$53 \cdot 431 \cdot a$, $23327 \cdot a$ ($a = 2^8 \cdot 3^2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 37 \cdot 73$)

$13 \cdot 23 \cdot 1109 \cdot b$, $71 \cdot 5179 \cdot b$ ($b = 2^{15} \cdot 3^4 \cdot 11^3 \cdot 17 \cdot 31 \cdot 43 \cdot 61 \cdot 257$) and for the same value of b

$13 \cdot 23 \cdot 251 \cdot b$, $97 \cdot 863 \cdot b$ and $13 \cdot 23 \cdot 149 \cdot b$, $199 \cdot 251 \cdot b$

$7 \cdot 59 \cdot c$, $19 \cdot 23 \cdot c$ ($c = 2^7 \cdot 3^3 \cdot 5$)

$7 \cdot 53 \cdot d$, $17 \cdot 23 \cdot d$ ($d = 2^6 \cdot 3^3 \cdot 5$)

$43 \cdot 167 \cdot 449 \cdot e$, $59 \cdot 131 \cdot 419 \cdot e$ ($e = 2^7 \cdot 3 \cdot 5^2 \cdot 7 \cdot 31$)

$7 \cdot 89 \cdot f$, $23 \cdot 29 \cdot f$ ($f = 2^{13} \cdot 3^3 \cdot 5 \cdot 127$)

$59 \cdot 599 \cdot g$, $71 \cdot 499 \cdot g$ ($g = 2^{20} \cdot 3 \cdot 5^2 \cdot 7 \cdot 31 \cdot 127$).

A multiply amicable triple with $t = 2$ is given by $59 \cdot 331 \cdot a$, $23 \cdot 829 \cdot a$, $19919 \cdot a$, where a is found by omitting the factor 47 from a multiply perfect number of multiplicity 6 and using the remainder of that number as a, provided that the multiply perfect number does not contain any of the factors 23, 59, 331, 829, 19919.

Conclusion. The methods of finding amicable number sets are very largely those of trial. Experience in working with such numbers will suggest the likely numbers to try, but there is no sure guide yet known. The number of cases that need to be tried out becomes very large for some of the larger numbers that we might choose for $n_1, n_2, \cdots, n_k$ in seeking k-tuples of the form $an_1, an_2, \cdots, an_k$. For example, the number 1108800 is the sum of the divisors of at least twenty different sets of prime factors no one of which is less than 19. If we are using the method suggested by Dickson and are seeking amicable sextuples we shall have as many possibilities as there are combinations of twenty, six at a time. The same set offers ${}_{20}C_2$ possibilities for amicable pairs, ${}_{20}C_3$ for triples, etc. Any systematic search for amicable numbers among the large numbers will furnish a vast amount of work.

The author of this paper was attracted to the subject on reading the section

[1] Carmichael and Mason, *Proceedings of the Indiana Academy of Science*, 1911, pp. 257–270.

[2] Indicated but not explicitly stated by Carmichael, in this MONTHLY, *1919*, 399.

on amicable numbers in Dickson's *History of the Theory of Numbers* and seeing how large a number of papers had been written on the subject and how relatively few numbers had been found. After some time spent in searching he is convinced that any one with a little skill in manipulation of numbers, considerable patience, and access to the Lehmer's *List of Prime Numbers*, can add to the known list of amicable number pairs, amicable k-tuples or multiply amicable number sets.

AN ELEMENTARY TREATMENT OF FOURIER'S SERIES.

By GEORGE D. BIRKHOFF,[1] Harvard University.

The aim of this note is to treat the "remainder" after $n+1$ terms of the Fourier's series for a given function $f(x)$:

$$\tfrac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots,$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x) \cos kx dx, \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x) \sin kx dx.$$

It will be assumed that the function $f(x)$ is periodic of period 2π and continuous together with its first three derivatives.

Let $T_n(x)$ stand for the sum of the first $n+1$ terms of the above series; its derivative $T_n'(x)$ is seen at once not to exceed in numerical value

$$(|a_1| + |b_1|) + 2(|a_2| + |b_2|) + \cdots + n(|a_n| + |b_n|).$$

Furthermore, by integration by parts and use of the periodicity of $f(x)$, we find

$$a_k = -\frac{1}{\pi k}\int_{-\pi}^{+\pi} f'(x) \sin kx dx = -\frac{1}{\pi k^2}\int_{-\pi}^{+\pi} f''(x) \cos kx dx$$

$$= \frac{1}{\pi k^3}\int_{-\pi}^{+\pi} f'''(x) \sin kx dx.$$

Consequently $|a_k|$, and similarly $|b_k|$, is less than $2F_3/k^3$ where F_3 is the maximum of $|f'''(x)|$. We conclude that $|T_n'(x)|$ is not greater than

$$4F_3\left(\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right) < 4F_3\left(1+\left(1-\frac{1}{2}\right)+\cdots+\left(\frac{1}{n-1}-\frac{1}{n}\right)\right),$$

[1] Professor Birkhoff's earlier paper in this MONTHLY, "Note on certain quadratic number systems for which factorization is unique," appeared almost exactly fifteen years ago. His first published paper, in collaboration with H. S. Vandiver, "On the integral divisors of $a^n - b^n$," was published in *Annals of Mathematics*, 1904. His second paper appeared also in *Annals . . .*, 1905. His third paper was a thirty page memoir in *Transactions of the American Mathematical Society*, 1906; and his fourth paper is the one referred to above. A complete collection of his mathematical papers 1904–1919, 37 in number, is preserved in a bound volume at the mathematical seminary of Brown University.—EDITOR.

so that

$$|T_n'(x)| < 8F_3. \tag{1}$$

The expression $T_n(x)$ has also the properties

$$\begin{cases} a_k = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{+\pi} T_n(x)\cos kx dx, \\ b_k = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{+\pi} T_n(x)\sin kx dx, \end{cases} \tag{2}$$

for $k \leqq n$, as follows by direct integration. Using the equations of definition of a_k, b_k in combination with (2) we deduce at once

$$\int_{-\pi}^{+\pi} R_n(x)\cos kx dx = \int_{-\pi}^{+\pi} R_n(x)\sin kx dx = 0, \qquad (k \leqq n), \tag{3}$$

where $R_n(x)$ denotes the "remainder" $f(x) - T_n(x)$.

Now by means of the familiar product formulas of trigonometry we may express successively $\cos^2 x$, $\cos x \sin x$, $\sin^2 x$, $\cos^3 x$, $\cdots$, $\sin^n x$ as linear combinations of 1, $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cdots$, $\sin nx$. Therefore we find from (3)

$$\int_{-\pi}^{+\pi} R_n(x)\cos^\alpha x \sin^\beta x dx = 0, \qquad (\alpha + \beta \leqq n), \tag{3'}$$

and, noting the equation

$$\cos^2\frac{x - x_0}{2} = \tfrac{1}{2}(1 + \cos x \cos x_0 + \sin x \sin x_0),$$

we infer from (3')

$$\int_{-\pi}^{+\pi} R_n(x)\cos^{2n}\frac{x - x_0}{2} dx = 0 \tag{4}$$

for any x_0. For convenience we make the change of variables $x = x_0 + t$ and obtain

$$\int_{-\pi}^{+\pi} R_n(x_0 + t)\cos^{2n}\frac{t}{2} dt = 0. \tag{4'}$$

The limits can be made $\pm\pi$ since the integrand is periodic of period 2π in t.

By the aid of (4') we propose to obtain an upper limit for the "remainder" $R_n(x_0)$. We have

$$\begin{aligned} |R_n(x_0 + t) - R_n(x_0)| &\leqq |f(x_0 + t) - f(x_0)| + |T_n(x_0 + t) - T_n(x_0)| \\ &\leqq (F_1 + 8F_3)|t|, \end{aligned}$$

where F_1 stands for the maximum of $|f'(x)|$ and $8F_3$ is at least as great as

$|T_n'(x)|$ by (1). Now the ratio

$$\frac{y}{\sin y} \qquad \left(|y| \leqq \frac{\pi}{2}\right)$$

is positive and may be seen geometrically or otherwise to have its maximum value $\pi/2$ for $y = \pm \pi/2$. Therefore it is clear that

$$|t| \leqq \pi \left|\sin \frac{t}{2}\right|, \qquad (|t| \leqq \pi),$$

whence we find

$$|R_n(x_0 + t) - R_n(x_0)| \leqq N|\sin \tfrac{1}{2}t|, \quad (N = \pi(F_1 + 8F_3)). \tag{5}$$

Thus we may substitute in (4')

$$R_n(x_0 + t) = R_n(x_0) + \theta N \sin \tfrac{1}{2}t \tag{5'}$$

where $|\theta| \leqq 1$. When we do so there results

$$R_n(x_0) = -\frac{\int_{-\pi}^{+\pi} \theta N \cos^{2n}\frac{t}{2} \sin\frac{t}{2}\, dt}{\int_{-\pi}^{+\pi} \cos^{2n}\frac{t}{2}\, dt}. \tag{6}$$

The numerator on the right in (6) is not greater numerically than

$$N\int_{-\pi}^{+\pi} \cos^{2n}\frac{t}{2}\left|\sin\frac{t}{2}\right| dt = 2N\int_0^{\pi} \cos^{2n}\frac{t}{2}\sin\frac{t}{2}\, dt = \frac{4N}{2n+1}.$$

Also the familiar formulas

$$\int_0^{\pi/2} \cos^n x\, dx = \begin{cases} \dfrac{1\cdot 3\cdots n-1}{2\cdot 4\cdots n}\cdot\dfrac{\pi}{2} & (n \text{ even}), \\ \dfrac{2\cdot 4\cdots n-1}{3\cdot 5\cdots n} & (n \text{ odd}), \end{cases}$$

show that

$$\int_0^{\pi/2} \cos^n x dx \cdot \int_0^{\pi/2} \cos^{n+1} x dx = \frac{\pi}{2(n+1)}.$$

Of the two integrals on the left the first is of course the greater, and it follows that

$$\int_0^{\pi/2} \cos^n x dx > \sqrt{\frac{\pi}{2(n+1)}}, \qquad (n = 0, 1, \cdots),$$

and thus that

$$\int_{-\pi}^{+\pi} \cos^{2n}\frac{t}{2}\, dt = 4\int_0^{\pi/2} \cos^{2n} x dx > 2\sqrt{\frac{2\pi}{2n+1}}.$$

If we replace the numerator and denominator in (6) by the greater and lesser values respectively which have now been found, the fraction is increased numer-

ically; whence finally

$$|R_n(x)| < N\sqrt{\frac{2}{\pi(2n+1)}} \tag{7}$$

for every value x_0 of x and for every $n \geqq 0$. Thus the remainder approaches 0 as n increases indefinitely, and the Fourier's series converges to $f(x)$.

The method outlined above may easily be modified so as to deal with a more general type of function $f(x)$, for instance any periodic function with two derivatives. Essentially the only novel feature of the method is the direct use of an equation like (4) to obtain an upper limit for the remainder.[1]

A CUBIC SPACE CURVE CONNECTED WITH THE TETRAHEDRON.

By FRANCIS D. MURNAGHAN, Johns Hopkins University.

In connection with the geometry of the triangle there is a well known circumconic, known as Kiepert's hyperbola,[2] which passes through the orthocenter and centroid of the triangle. In the following note an analogue of this hyperbola for the tetrahedron is given, and it is hoped that interest may thereby be aroused in the still comparatively unexplored field of the "geometry of the tetrahedron." [3]

It will be convenient to recall an interesting property of Kiepert's hyperbola—a property which may be used to define the curve and at the same time to give a simple parametric representation of the points on it. Let A_1, A_2, A_3 be the triangle and on each of the segments A_2A_3, A_3A_1, A_1A_2 describe a circle containing an angle θ; θ being in each case the angle subtended by the segment at points of that arc of the corresponding circle which is on the same side of the segment as the opposite vertex, so that θ can vary from 0° to 180°. For any given value of θ the three circles have a radical center and Kiepert's hyperbola is the locus of these radical centers. In order to find the coördinates of the radical center for a given value of θ, it is probably simplest to introduce, momentarily, rectangular axes with origin at O_1, the mid-point of A_2A_3, and with the X axis along A_2A_3. If $2a_1$ is the length of A_2A_3, the coördinates of C_1, the center of the circle through A_2A_3, are $(0,\ a_1 \cot\theta)$ and the radius is $a_1 \operatorname{cosec}\theta$, so that the equation is

[1] See Lebesgue, *Leçons sur les séries trigonométriques*, Paris, 1906, pp. 37–38, and de la Vallée Poussin, *Bulletins de l'Académie royale de Belgique*, classe des sciences, 1908, pp. 193–254, in particular p. 230 *et seq.*

[2] Kiepert, *Nouvelles Annales de Mathématiques*, 1869, p. 42. Other discussions of this hyperbola are: by Brocard, in *Journal de Mathématiques Spéciales*, 1884, pp. 197–209 and 1885, pp. 12–15, 30–33, 58–64, 76–80, 104–112, 123–131; by de Longchamps, *idem*, 1886, pp. 77–79 231–235; by M'Cay, in *Mathesis*, 1887, pp. 208–220; by Laisant, in *Compte rendu . . . Association Française pour l'Avancement des Sciences*, seconde partie, 1887, pp. 113–114; by J. Casey, in his *A Treatise on the Analytical Geometry of the Point, Line, Circle and Conic Sections*, second edition, 1893, pp. 431, 442–445, 449, 452, 453; and by I. J. Schwatt, in his *A geometrical Treatment of Curves which are isogonal conjugate to a straight line with respect to a triangle*. Part 1, Boston, New York, and Chicago, [c. 1895], pp. 3–28.—EDITOR.

[3] Those interested may be referred to two papers by J. Neuberg in *Archiv der Mathematik und Physik*, 3. Reihe, vols. 16 and 18, 1910–1911.

$X^2 + Y^2 - 2a_1 Y \cot\theta - a_1^2 = 0$. The equation of the circumcircle is

$$S \equiv X^2 + Y^2 - 2a_1 Y \cot A_1 - a_1^2 = 0,$$

so that our circle may be written in the form $S + 2a_1 Y(\cot A_1 - \cot\theta) = 0$. It will now be better to introduce trilinear coördinates (x_1, x_2, x_3), x_1 signifying as usual the perpendicular distance of the point from the side A_2A_3, and so being identical with Y above. Accordingly our equation is $S + 2a_1x_1(\cot A_1 - \cot\theta) = 0$ where $S = 0$ is the equation of the circumcircle in trilinear coördinates (so written that its norm—the coefficient of $X^2 + Y^2$ when we change over to Cartesian coördinates—is unity). The equations of the other two circles are, similarly,

$$S + 2a_2x_2(\cot A_2 - \cot\theta) = 0, \qquad S + 2a_3x_3(\cot A_3 - \cot\theta) = 0;$$

and hence the radical center is given by the equations

$$a_1x_1(\cot A_1 - \cot\theta) = a_2x_2(\cot A_2 - \cot\theta) = a_3x_3(\cot A_3 - \cot\theta).$$

Or, since $\cot A_1 - \cot\theta = \sin(\theta - A_1)/\sin A_1 \sin\theta$, and $a_1/\sin A_1 =$ radius of circumcircle,

$$x_1 \sin(A_1 - \theta) = x_2 \sin(A_2 - \theta) = x_3 \sin(A_3 - \theta),$$

whence

$$x_1 : x_2 : x_3 = \frac{1}{\sin(A_1 - \theta)} : \frac{1}{\sin(A_2 - \theta)} : \frac{1}{\sin(A_3 - \theta)}.$$

If we wish to find the locus of the radical center we have

$$k/x_1 = \sin(A_1 - \theta) = \sin A_1 \cos\theta - \sin\theta \cos A_1,$$

and two similar equations where k is a factor of proportionality. These being linear and homogeneous in k, $\cos\theta$, $\sin\theta$ we have

$$\begin{vmatrix} \frac{1}{x_1} & \sin A_1 & \cos A_1 \\ \frac{1}{x_2} & \sin A_2 & \cos A_2 \\ \frac{1}{x_3} & \sin A_3 & \cos A_3 \end{vmatrix} = 0,$$

or expanded,

$$\frac{1}{x_1}\sin(A_2 - A_3) + \frac{1}{x_2}\sin(A_3 - A_1) + \frac{1}{x_3}\sin(A_1 - A_2) = 0;$$

and finally,

$$x_2x_3 \sin(A_2 - A_3) + x_3x_1 \sin(A_3 - A_1) + x_1x_2 \sin(A_1 - A_2) = 0.$$

It is immediately verifiable that this circumconic goes through the orthocenter

(sec A_1, sec A_2, sec A_3) and the centroid (cosec A_1, cosec A_2, cosec A_3), but it is better to go back to the parametric equations of the curve. $\theta = 0°$ gives the centroid; $\theta = 90°$ gives the orthocenter. In addition $\theta = 60°$, $\theta = 120°$ give the two *Fermat Points, i.e.*, the sum of the distances of the point $\theta = 120°$ to the three vertices $A_1\ A_2\ A_3$ is a minimum.[1] It is left as an interesting exercise to prove that if on each segment A_2A_3, A_3A_1, A_1A_2 *isosceles* triangles with base angles θ are described such that when $0 < \theta < 90°$, each of the triangles has its third vertex on the *same* side of the segment as the remaining vertex of the original triangle, and when $90° < \theta < 180°$ on the *opposite* side, the joins of each vertex A to the vertex of the corresponding isosceles triangle are concurrent at the radical center discussed[2] above. In particular when the triangles are equilateral, *i.e.*, when $\theta = 60°$ or $120°$ we have the usual construction for the Fermat points.

It is now easy to follow the extension to the tetrahedron. Here we have four vertices A_1, A_2, A_3, A_4; four faces with areas a_1, a_2, a_3, a_4, circumcenters O_1, O_2, O_3, O_4 and circumradii R_1, R_2, R_3, R_4. What is to take the place of the circles of angle θ? A natural analogue of the circle 1 is a sphere through A_2, A_3, A_4 with the center at C_1 such that the angles $A_2C_1O_1 = A_3C_1O_1 = A_4C_1O_1 = \theta$. To find the equation of this sphere introduce, for the moment, rectangular axes (X, Y, Z) with origin at O_1 and Z axis along O_1C_1. The coördinates of the center C_1 are $(0,\ 0,\ R_1 \cot\theta)$, and the radius is $R_1 \operatorname{cosec}\theta$, so that the equation is $X^2 + Y^2 + Z^2 - 2R_1Z\cot\theta - R_1^2 = 0$. Now let A_1 denote the angle θ for the circumsphere so that the equation of the latter is $S \equiv X^2 + Y^2 + Z^2 - 2R_1Z\cot A_1 - R_1^2 = 0$ and we have as before, on introducing for the sake of symmetry tetrahedral coördinates (x_1, x_2, x_3, x_4), that the equations of the four spheres are $S + 2x_1R_1(\cot A_1 - \cot\theta) = 0$, etc., S being the equation of the circumsphere so written that its norm is unity. The radical center is accordingly given by

$$x_1 : x_2 : x_3 : x_4 = \frac{1}{\sin(A_1 - \theta)} : \frac{1}{\sin(A_2 - \theta)} : \frac{1}{\sin(A_3 - \theta)} : \frac{1}{\sin(A_4 - \theta)},$$

where we have availed ourselves of the fact that $R_1 = R \sin A_1$, R being the radius of the circumsphere. This is a curve through the vertices of the tetrahedron of reference; for when $\theta = A_1$ we get the vertex $x_1 : x_2 : x_3 : x_4 = 1 : 0 : 0 : 0$. Just as Kiepert's hyperbola degenerates into a line when the triangle is isosceles so this space curve will degenerate into a plane curve if any of the angles $A_1\ A_2\ A_3\ A_4$ are equal. It is at once apparent that an easy way to study the curve is to make use of the *focal transformation*[3]

$$y_1 = \frac{1}{x_1}; \qquad y_2 = \frac{1}{x_2}; \qquad y_3 = \frac{1}{x_3}; \qquad y_4 = \frac{1}{x_4}.$$

[1] Compare this MONTHLY, *1920*, 38–41.

[2] Kiepert, *l.c.*—EDITOR.

[3] This Cremona Transformation is of frequent occurrence and is called "inversion" by the French writers. However most mathematicians will agree that this word is rather overworked and the name used in the text is suggested. The points (x), (y) are the two foci of a quadric of revolution inscribed in the tetrahedron of reference.

Then the point y traces out a *line* which is the intersection of the two planes,

$$y_1 \sin (A_2 - A_3) + y_2 \sin (A_3 - A_1) + y_3 \sin (A_1 - A_2) = 0,$$

$$y_2 \sin (A_3 - A_4) + y_3 \sin (A_4 - A_2) + y_4 \sin (A_2 - A_3) = 0.$$

Now the transformation $y = 1/x$ sends an arbitrary plane into a cubic surface which meets a line in three points, and accordingly an arbitrary plane meets our curve in three points so that it is a *cubic* space curve. The point $\theta = 0$, which in the plane case gave the centroid, here gives the point $x_1 : x_2 : x_3 : x_4 = \operatorname{cosec} A_1 : \operatorname{cosec} A_2 : \operatorname{cosec} A_3 : \operatorname{cosec} A_4 = (1/R_1) : (1/R_2) : (1/R_3) : (1/R_4)$. The point $\theta = 90°$, which previously gave the orthocenter, is here $x_1 : x_2 : x_3 : x_4 = \sec A_1 : \sec A_2 : \sec A_3 : \sec A_4$. This point I propose to call the *Roberts's point*, since S. Roberts, in the *Educational Times*, 1887, problem 9093,[1] stated that the radical center of the four spheres, described on the faces as diametral planes, is the focal transform of the circumcenter of the tetrahedron. The straight line traced by the point y is accordingly described by saying that it passes through the circumcenter of the tetrahedron and the point whose coördinates are proportional to R_1, R_2, R_3, R_4.

That the curve whose parametric equations are

$$x_i = \operatorname{cosec} (A_i - \theta), \qquad i = 1, 2, 3, 4,$$

is a cubic can be seen easily in a more direct manner as follows: Eliminating θ from the three equations obtained by writing $i = 2, 3, 4$ in turn, we find

$$x_3x_4 \sin (A_3 - A_4) + x_4x_2 \sin (A_4 - A_2) + x_2x_3 \sin (A_2 - A_3) = 0.$$

This is a quadric cone with its vertex at the point $(1, 0, 0, 0)$ and passing through the other vertices. Similarly the curve lies on the quadric cone

$$x_3x_4 \sin (A_3 - A_4) + x_4x_1 \sin (A_4 - A_1) + x_1x_3 \sin (A_1 - A_3) = 0.$$

These cones have a common generator—the join of the vertices A_1A_2 of the tetrahedron, and the remaining intersection is a cubic curve.

In conclusion one remark further may be made. It is a classical theorem on cubic space curves that, given four points on one, the cross ratio of the four planes, one through each of the points and a variable secant, is constant.[2] Taking the four vertices of the tetrahedron as the four fixed points, and the secant as passing through the *Roberts's* point and the point $(1/R_1, 1/R_2, 1/R_3, 1/R_4)$, it is not difficult to see that this cross ratio is

$$\sin (A_1 - A_3) \sin (A_2 - A_4)/[\sin (A_1 - A_4) \sin (A_2 - A_3)].$$

[1] *Mathematical Questions and Solutions from the "Educational Times,"* vol. 48, 1883, pp. 37, 179.—EDITOR.

[2] Pascal, *Repertorium der höheren Mathematik*, vol. 2, 1902, p. 250.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

6. The Threatened Loss of the Second Edition of Montucla's History of Mathematics.

The period of the French Revolution was not favorable to scientific work. One has only to read M. Bigourdan's delightfully written history of the metric system, or to consider the quasi-judicial murder of Bailly and the tragic suicide of Condorcet, in order to understand the hostile attitude of the *sansculottes* to all intellectual pursuits.

It was during this period that Montucla (1725–1799), then upwards of seventy years of age, interested the astronomer La Lande (1732–1807) to prepare the second edition of the *Histoire des Mathématiques* which had appeared nearly half a century earlier. The learned world knows how La Lande, with the aid of various other distinguished scholars, carried out the undertaking, but it has long since forgotten, if it ever knew, the discouragements which were met by both Montucla and his more active colleague before the work finally made its appearance in the years 1799–1802.

Among the most interesting brief notes in my collection of nearly forty autograph letters of La Lande is one which he wrote on the occasion of a call upon M. Henri Agasse, in the year 1796, three years before the first volume of this edition appeared. M. Agasse was the publisher, his establishment being at No. 18, rue des Poitevins. La Lande was evidently unsuccessful in finding the publisher in, and so he took a scrap of paper and wrote upon it, in a few lines, and in his well-known microscopic hand, the following interesting story:

5 Thermidor an 4 [July 23, 1796]

Lalande has come to solicit Citizen Agasse with respect to continuing the history of mathematics. Montucla is disgusted, annoyed, at the interruption, on the point of being tempted to abandon the entire work and of burning the whole thing up. Have pity on an old invalid.

The solicitation, however, seems to have produced but little result, for it was not until three years later that the first volume appeared, only four months before Montucla passed away.

The other letters of La Lande throw an interesting side light upon the history of the time in which he lived. From only about forty letters it is, of course, impossible to generalize with a high degree of certainty upon such a trivial matter as the spelling of a name, and moreover it is well known that little attention was paid to such details before the nineteenth century. One interesting thing is apparent, however, and that is the effect of the French Revolution, and especially the Reign of Terror, upon the use of aristocratic names in general.

In 1755, when La Lande was twenty-three years old he wrote his name Lalande. Youth leans to democratic forms. As he grew older he used the form

La Lande, this occasionally appearing in letters between 1770 and 1786, but he preferred the more aristocratic De la Lande. As soon as the Reds came into power, however, he dropped the De and wrote only Lalande. This form he continued to use until Napoleon made his spectacular appearance and began to look forward to occupying the imperial throne. Then La Lande returned again to the form De la Lande, which he had preferred before the Revolution, and this form, which also appears on the title pages of volumes II and III, he seems to have generally used until the time of his death, in 1807.

Undoubtedly a similar course was taken by all upon whom ancestral rights conferred the coveted *de*, but it may interest those whose work leads them to consider the contributions of La Lande to have this information from a rather large collection of his letters. The matter is of little consequence, and would have even less importance if biographers were more careful in examining the correspondence of those concerning whom they write; but everyone who has occasion to consult the various monographs and books relating to the lives of scientists knows that such details have little attention on the part of writers. On this account the information may be of some slight service.

7. Monge the Lesser.

It often happens, in the reading of history, that individuals who would naturally stand out with some prominence are obscured by the overshadowing of identical names. Thus we have the two Āryabhaṭas in India, the two Psellus's in Byzantium, the two al-Khowârizmîs in Bagdad, and the two Aragos in France, each case giving rise to some confusion to the casual reader, and the lesser of the two being obscured by the light of the greater.

One of the letters which I have and which bears the name of Monge illustrates this fact. It is in itself of little consequence, but if Gaspard Monge had not been the genius that he was, his brother Louis might have been remembered. The letter is as follows:

> Toulon, 20 Nivose, year 5 [January 9, 1797] Monge, Examiner in hydrography in the Navy, to Citizen Groignard, Ordonnateur [director] of the Navy.
>
> *Citizen Ordonnateur,*
>
> The minister having authorized you, by his letter of the 26th thermidor, to give me such pecuniary aid as I may need on my journey, I beg you to be so good as to remit to me the sum of 950# [i.e., 950 French livres], which will be necessary not only to pay for my sojourn at Marseilles and at Toulon, but also for my return to Paris.
>
> Salut et fraternité,
>
> Monge.

In his eloquent eulogy on Gaspard Monge, delivered in 1846, Arago speaks of the struggles of Jacques Monge, the father, to obtain the means of sending his three sons to the college at Beaune, their native city. He records how the three sons "responded to the paternal solicitude," each making a success in his profession, Gaspard, however, being the only one to achieve fame.

The writer of the above letter was Louis Monge. He was born on April 11, 1748, and died on October 6, 1827. He was therefore two years younger than

his illustrious brother and survived him by nine years. He was well and favorably known both as a mathematician and as a hydrographer. He was attached to the ill-fated voyage of discovery led by La Pérouse (La Peyrouse), but fortunately was compelled to abandon the trip owing to illness which developed at Teneriffe. He became professor of mathematics in the Ecole militaire, and later became professor and examiner in hydrography in the navy. Thanks to his brother's influence, the latter having a position as minister in the revolutionary period, he passed safely through the Days of Terror, and in 1797, when the worst was over, he occupied the position which the letter indicates.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

The discussion by Professor Moritz on geometric illustrations of indeterminate forms should prove interesting and valuable. It is not superfluous to take this occasion to point out the extraordinary misunderstanding and logical inaccuracy often associated with the subject of indeterminate forms. Errors are common in our textbooks, and some teachers have never become acquainted with the correct point of view. It will be necessary first to touch upon the ideas of zero and infinity, which are scarcely less hazy.

In the first place, it should be clearly understood that the number system which is agreed upon by mathematicians as best representing the practical needs of analysis—irrespective of the obvious possibility of other self-consistent number systems, whose properties may agree to a greater or less extent with those of the usual system—does contain a number *zero*, as real, with as definite a meaning as any other number, and having, with one exception, properties like those of any other number. This ordinary number system, on the other hand, does *not* contain a number *infinity*. If any one prefers a system which shall not contain zero or which shall contain infinity, then, so long as he formulates self-consistent laws of operation, he is entirely free to use his system; but he cannot expect mathematicians generally to follow his lead unless he can show that his system involves fewer exceptions or inconveniences than that to which they are accustomed.

If we are to use the ordinary system, we shall have no difficulty provided we take account of the fact that its laws are uniform and general, except in one single particular: the number zero cannot be used as a divisor. The expression $2/0$ does not represent some mysterious unattainable large number; it represents no number whatever. The expression $0/0$ does not represent several, many, or all numbers[1]; it represents no number whatever.

[1] A logical system might be set up in which $0/0$ would have this multiple-valued interpretation; such a system would apparently be about as convenient as the usual one; but it is not the usual one.

In what way, then, is the statement

$$\frac{2}{0} = \infty$$

to be interpreted? Strictly, in an arithmetical sense, it signifies merely that the operation $2/0$ is meaningless;—that the "result" of the operation is of one kind with the unicorn and the sea serpent. It is also used in a sense not exclusively arithmetical, generally with some sacrifice in clearness, to mean that "as x approaches zero, $2/x$ becomes infinite," or, in other words, "$2/x$ can be made as large as we please by taking x sufficiently small." It should be noted that even this interpretation implies no division by zero, but does imply division by many other numbers,—numbers near zero.

When we consider the fraction $(x^2 - 1)/(x - 1)$ for the value $x = 1$, we find that both numerator and denominator are entirely respectable numbers,—both are in fact zero; the fraction however is meaningless, since division by zero is a meaningless operation. When on the other hand we consider the fraction $(\tan \frac{1}{2}\pi x)/(\sec \frac{1}{2}\pi x)$ for $x = 1$, we find that even the numerator and denominator individually are meaningless.

But in both cases, when we speak of "evaluating the indeterminate form," we mean (or ought to mean) something which is unfortunately quite different from the literal implication of the words. We do not mean that we are to discover, by astute detective work, a "true value" which unkind fate is endeavoring to conceal from us. It is in no sense true that $(x^2 - 1)/(x - 1)$, which for $x = 1$ assumes the form $0/0$, really takes on for $x = 1$ the "true value" or "determinate value" 2. The value of $(x^2 - 1)/(x - 1)$ for $x = 1$ is not 2, does not exist, and cannot be brought into existence unless some other than the usual number system be used. What we can do, and what is often useful for us to do, is to find the limit approached by $(x^2 - 1)/(x - 1)$ as x approaches 1; this limit is indeed 2. If we write, as do some text-books:

$$\text{"(1)} \quad \frac{x^2 - 1}{x - 1} = x + 1;$$

hence

$$\text{(2)} \quad \frac{x^2 - 1}{x - 1} = 2 \qquad \text{when} \qquad x = 1,\text{"}$$

we are merely confusing the issue by a bald untruth. The following statement is true, and is the most that can be said without departing from the truth:

$$\text{"(1)} \quad \frac{x^2 - 1}{x - 1} = x + 1 \qquad \text{when} \qquad x \neq 1;$$

hence

$$\text{(2)} \quad \frac{x^2 - 1}{x - 1} \text{ approaches 2 as } x \text{ approaches 1."}$$

As indicated, (1) is true for $x \neq 1$. It is untrue for $x = 1$; the algebraic

operation by which it was obtained—formal division of the numerator by the denominator—cannot be performed when $x = 1$, although it can be performed for every other value of x.

Similarly, when we set out to "evaluate the indeterminate form $(\tan \frac{1}{2}\pi x)/(\sec \frac{1}{2}\pi x)$ for $x = 1$," we must not think that we shall by sharpness of vision pierce an algebraic mist which hides from us the "true value" 1 of this expression for $x = 1$. We should mean that we are seeking the limit (if any exists) approached by $(\tan \frac{1}{2}\pi x)/(\sec \frac{1}{2}\pi x)$ as x approaches 1; and we should mean nothing further than this.

For the value $x = 1$, $(x^2 - 1)/(x - 1)$ does assume the form $0/0$, and $(\tan \frac{1}{2}\pi x)/(\sec \frac{1}{2}\pi x)$ may be said to assume the form ∞/∞; but neither fraction has any *value* for the value of x in question, nor can we by any amount of ambiguous argument or inaccurate analogy force it to have such a value.

On The Geometrical Representation of Indeterminate Forms.

By R. E. Moritz, University of Washington.

Every teacher of the calculus will have witnessed the difficulty beginners have in properly interpreting the so-called indeterminate forms $0/0$, ∞/∞, $0 \cdot \infty$, $\infty - \infty$, etc. This should cause no surprise; for the meaning of the symbols 0 and ∞, even when taken by themselves, is not too clear to the average student and the difficulty is, of course, accentuated when these symbols are combined as in the indeterminate forms.

It should be remembered in this connection that the indeterminate forms have not always been correctly interpreted even by mathematicians and that some of them remained subjects of controversy until comparatively recent times. The first complete evaluation of the form $0/0$ dates from John Bernoulli[1] (1704) and it was not until 1755 that Euler[2] gave the first published treatment of the forms ∞/∞, $0 \cdot \infty$, and $\infty - \infty$. But the true meaning of 0^0 eluded even the great Euler, for he arrived at the conclusion[3] that 0^0 must always equal 1 by a process of reasoning which would show equally well that $0/0$ must always equal 1. Euler's erroneous conclusion was shared by many subsequent mathematicians for three quarters of a century. New proofs (!) of his conclusion appeared in creditable journals by eminent mathematicians as late as the middle third of the nineteenth century,[4] and this notwithstanding the great weight of the authority of Cauchy, who, in his *Cours d'Analyse*,[5] had given the interpretation of the exponential indeterminate forms which is universally accepted today.

In view of the fact that modern textbooks on the calculus generally avail themselves of geometrical illustrations to introduce new or difficult concepts, it

[1] *Opera*, vol. 1, pp. 401–405.
[2] *Opera Omnia*, series I, vol. 10, 1913, p. 576.
[3] Euler, *Opera Omnia*, series I, vol. 1, 1911, p. 65.
[4] Libri, G., Crelle's *Journal*, vol. 10, 1833, p. 303; Moebius, A. F., Crelle's *Journal*, vol. 12, 1834, p. 134.
[5] *Oeuvres*, 2d series, vol. 3, p. 70.

seems unfortunate that such aids are not used in introducing the concept of indeterminate forms. In this connection geometrical illustrations are so conspicuously absent that the cynically inclined reader might suspect writers of a conspiracy to mystify rather than to clarify the subject under discussion. With one notable exception[1] in the case of a text long out of use (and now out of print) the writer does not know of a single textbook in either English, French, or German, which so much as hints at the existence of geometrical illustrations in treating the subject of indeterminate forms, although every modern text, so far as the writer is aware, freely uses geometrical aids in introducing the concepts of the derivative and the definite integral.

Lovitt,[2] in a short paper on the subject, gave five illustrations of indeterminate forms chosen from the fields of solid geometry, trigonometry, the geometry of the triangle, the geometry of the circle, and hydraulics respectively. These illustrations lack, however, the desired perspicuity and, in the majority of cases, are faulty in two respects: first, in that the variables actually reach their limits, thus rendering the expressions which give rise to the indeterminate forms meaningless; and second, in that the ratio, product, or difference which assumes the indeterminate form is constant and thus fails to illustrate the really important case of a function of x which approaches a limiting value but an indeterminate form as the variable approaches some definite limit.

It is the purpose of the present paper to exhibit some simple plane-geometric illustrations of the elementary indeterminate forms. Care has been exercised that in each case the function which in the limit assumes an indeterminate form is really a function of the variable assuming different values and not a constant in disguise. Moreover, the constructions are so conditioned that the question of dividing zero by zero, or infinity by infinity, cannot arise since the quantities considered are never zero or infinite. All our ratios are those of finite quantities which may be increased or decreased indefinitely but which, from the nature of their generation, can never be actually zero or infinite. The point here is that the problem of indeterminate forms may be so conceived that the question of dividing zero by zero, or infinity by infinity, which is the real source of confusion, need not arise.

Each of the figures that follow is so chosen that it illustrates at the same time each of the possible values zero, finite, infinite, which the indeterminate form under consideration may take. The last figure illustrates all the elementary indeterminate forms at once for the whole range of values which any of these forms may have. This figure is remarkable in that it interrelates one each of the four possible types of triangles, the ultimate values of whose sides are respectively

$$0, 0, 0; \qquad 0, a, b; \qquad a, \infty, \infty; \qquad \infty, \infty, \infty.$$

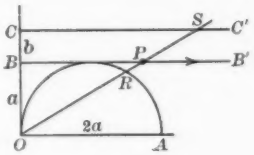
It would be interesting to know whether it is possible to construct a simpler figure that will serve the same purpose equally well or better.

[1] DeMorgan, A., *Differential and Integral Calculus*, Introductory, p. 19.
[2] American Mathematical Monthly, *1916*, 41.

Definitions. A variable distance is said to approach 0 as a limit, if it ultimately becomes, and remains, less than any preassigned distance however small.

A variable distance is said to approach a constant distance a as a limit if the difference between the constant distance and the variable distance ultimately becomes, and remains, less than any preassigned distance however small.

A variable distance is said to become infinite if it ultimately becomes, and remains, greater than any preassigned distance however large.



1. $\infty - \infty$. Consider the semicircle having $OA = 2a$ as a diameter. BB' is a tangent to the semicircle drawn parallel to OA, and CC' a line drawn parallel to BB' at a distance b from it on the side opposite from 0.

Let P be a point on BB' which moves indefinitely in the direction BB' at a uniform velocity. Join O to P and produce OP to meet CC' at S.

Now it is obvious that each of the variable distances OP, BP, RP, OS, ultimately becomes, and remains, larger than any preassigned distance however large, so that ultimately

$$OP - BP = \infty - \infty, \qquad OP - RP = \infty - \infty, \qquad OS - OP = \infty - \infty.$$

But

$$OP - BP = \frac{a^2}{OP + BP}, \text{ which approaches 0 as a limit,}$$

$$OP - RP = OR, \text{ which approaches } OA = 2a \text{ as a limit,}$$

$$OS - OP = PS, \text{ which becomes infinite;}$$

hence

$$OP \to \infty, \qquad BP \to \infty, \qquad RP \to \infty, \qquad OS \to \infty;$$

$$OP - BP \to 0, \qquad OP - RP \to 2a, \qquad OS - OP \to \infty.$$

If we take O as origin of coördinates, OA and OB as the directions of coördinate axes, and denote the coördinates of the point P by (x, a), we find

$$OP = \sqrt{x^2 + a^2}, \quad BP = x, \quad RP = (x - a)^2/\sqrt{x^2 + a^2}, \quad OS = \frac{a + b}{a}\sqrt{x^2 + a^2},$$

and as x becomes infinite we have

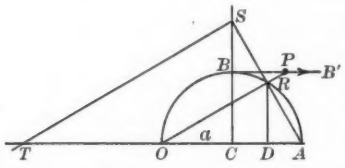
$$OP - BP = \sqrt{x^2 + a^2} - x = a^2/[\sqrt{x^2 + a^2} + x] \to 0,$$

$$OP - RP = \sqrt{x^2 + a^2} - (x - a)^2/\sqrt{x^2 + a^2} = 2ax/\sqrt{x^2 + a^2} \to 2a,$$

$$OS - OP = \frac{a + b}{a}\sqrt{x^2 + a^2} - \sqrt{x^2 + a^2} = \frac{b}{a}\sqrt{x^2 + a^2} \to \infty,$$

which verifies the preceding results analytically.

2. 0/0 and ∞/∞. Consider the semicircle having $OA = 2a$ as a diameter. BB' is the tangent to the semicircle drawn parallel to OA, P a point on it which moves indefinitely in the direction BB' at a uniform velocity.



Join P to O and denote the point in which OP intersects the semicircle by R. Join A and R and denote the point in which AR produced intersects the perpendicular CB to OA through the center C of the circle by S. Through S draw a perpendicular to AS and let this perpendicular intersect AO produced in T. Through R draw RD perpendicular to OA.

Now it is obvious that as P moves indefinitely along BB' in the direction of BB' each of the variable distances DA, AR, and RD ultimately becomes, and remains, less than any preassigned distance however small, and each of the variable distances CS, ST, and TC ultimately becomes, and remains, greater than any preassigned distance, however large, so that ultimately

$$\frac{DA}{RD} = \frac{0}{0}, \quad \frac{AR}{RD} = \frac{0}{0}, \quad \frac{RD}{DA} = \frac{0}{0}; \quad \frac{CS}{TC} = \frac{\infty}{\infty}, \quad \frac{ST}{TC} = \frac{\infty}{\infty}, \quad \frac{TC}{CS} = \frac{\infty}{\infty}.$$

But the triangles RDA, SCA, ODR, and TCS are similar, so that

$$\frac{DA}{RD} = \frac{CS}{TC} = \frac{DR}{OD}, \text{ which approaches 0 as a limit,}$$

$$\frac{AR}{RD} = \frac{ST}{TC} = \frac{RO}{OD}, \text{ which approaches 1 as a limit,}$$

$$\frac{RD}{DA} = \frac{TC}{CS} = \frac{SC}{CA}, \text{ which becomes infinite;}$$

hence

$$DA \to 0, \quad RD \to 0, \quad AR \to 0, \quad CS \to \infty, \quad TC \to \infty, \quad ST \to \infty;$$

$$\frac{DA}{RD} \to 0, \quad \frac{CS}{TC} \to 0, \tag{1}$$

$$\frac{AR}{RD} \to 1, \quad \frac{ST}{TC} \to 1, \tag{2}$$

$$\frac{RD}{DA} \to \infty, \quad \frac{TC}{CS} \to \infty. \tag{3}$$

We may obtain the same results analytically as follows. Take O as origin and OA as the direction of the x-axis of coördinates. Denote the coördinates of the point P by (x, a), then we find

$$DA = 2a^3/(x^2 + a^2), \quad AR = 2a^2/(\sqrt{x^2 + a^2}), \quad RD = 2a^2x/(x^2 + a^2),$$

$$CS = x, \quad ST = x\sqrt{x^2 + a^2}/a, \quad TC = x^2/a.$$

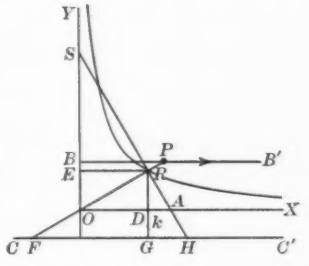
Substituting these values in the left members of (1), (2), (3), we have as x becomes infinite

$$\frac{DA}{RD} = \frac{2a^3/(x^2+a^2)}{2a^2x/(x^2+a^2)} = \frac{a}{x} \to 0, \qquad \frac{CS}{TC} = \frac{x}{x^2/a} = \frac{a}{x} \to 0;$$

$$\frac{AR}{RD} = \frac{2a^2/(\sqrt{x^2+a^2})}{2a^2x/(x^2+a^2)} = \frac{\sqrt{x^2+a^2}}{x} \to 1, \qquad \frac{ST}{TC} = \frac{x\sqrt{x^2+a^2}/a}{x^2/a} = \frac{\sqrt{x^2+a^2}}{x} \to 1;$$

$$\frac{RD}{DA} = \frac{2a^2x/(x^2+a^2)}{2a^3/(x^2+a^2)} = \frac{x}{a} \to \infty, \qquad \frac{TC}{CS} = \frac{x^2/a}{x} = \frac{x}{a} \to \infty.$$

3. $\infty \cdot 0$. Consider a branch of the equilateral hyperbola with $\sqrt{2}$ for the semi-major axis. Let O represent the center and OX, OY, the asymptotes to the hyperbola. Draw two parallels, BB' and CC', to one of the asymptotes on opposite sides of it and at distances a and k respectively.



Let P be a point on BB' which moves indefinitely in the direction of BB' at a uniform velocity. Join OP and denote its intersection with the hyperbola by R and its intersection with CC' by F. Through R draw a perpendicular to OR and denote its intersection points with OY, OX, and CC', by S, A, and H respectively. From R draw a perpendicular to OX, intersecting OX and CC' in D and G respectively. Through R draw a parallel to OX intersecting OY at a point E.

As the point P moves indefinitely in the direction BB' it is obvious that each of the variable distances FG, OD, ER, ES, ultimately becomes, and remains, greater than any preassigned distance however great, and each of the variable distances OE, DR, DA, GH, ultimately becomes, and remains, less than any preassigned distance however small, while GR approaches $GD = k$ as a limit; hence ultimately

$$OD \cdot DA = \infty \cdot 0, \qquad FG \cdot GH = \infty \cdot 0, \qquad ES \cdot OE = \infty \cdot 0.$$

But from the pairs of similar triangles ODR,RDA, FGR,RGH, SER,REO, we have

$OD \cdot DA = (DR)^2$, which approaches 0 as a limit,

$FG \cdot GH = (GR)^2$, which approaches k^2 as a limit,

$ES \cdot OE = (ER)^2$, which becomes infinite;

hence

$$OD \to \infty, \quad FG \to \infty, \quad ES \to \infty; \quad DA \to 0, \quad GH \to 0, \quad OE \to 0;$$

$$OD \cdot DA \to 0, \tag{4}$$

$$FG \cdot GH \to k^2, \tag{5}$$

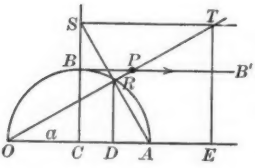
$$ES \cdot OE \to \infty. \tag{6}$$

To verify these results analytically, take O as the origin, OX as the x-axis of coördinates. Then, if we denote the coördinates of the point P by (x, a), we find

$$OD = \sqrt{(x/a)}, \qquad FG = (x/a)[k + \sqrt{(a/x)}], \qquad ES = \sqrt{(x/a)^3},$$
$$DA = \sqrt{(a/x)^3}, \qquad GH = (a/x)[k + \sqrt{(a/x)}], \qquad OE = \sqrt{(a/x)}.$$

On substituting these values in the left members of (4), (5), (6), we have as x becomes infinite,

$$OD \cdot DA = \sqrt{(x/a)} \cdot \sqrt{(a/x)^3} = a/x \to 0,$$
$$FG \cdot GH = (a/x)[k + \sqrt{(a/x)}] \cdot (x/a)[k + \sqrt{(a/x)}] = [k + \sqrt{(a/x)}]^2 \to k^2,$$
$$ES \cdot OE = \sqrt{(x/a)^3} \cdot \sqrt{(a/x)} = x/a \to \infty.$$



4. The figure illustrates all the preceding cases of indeterminate forms at the same time. The figure is like the one in § 2 with two additional lines, a line through S drawn parallel to OA and a perpendicular to OA through T, the point in which this line intersects OP produced.

As the point P moves indefinitely with a uniform velocity in the direction BB' the variable sides of the similar right triangles

$$RAD, \qquad OAR, \qquad SAC, \qquad TSR,$$

ultimately become or approach

$$0, 0, 0; \qquad 0, 2a, 2a; \qquad a, \infty, \infty; \qquad \infty, \infty, \infty.$$

We have then

$$AS - CS = \infty - \infty, \qquad OT - RT = \infty - \infty, \qquad OT - PT = \infty - \infty;$$
$$RS/ST = \infty/\infty, \qquad ST/RT = \infty/\infty, \qquad RT/RS = \infty/\infty;$$
$$CS \cdot DA = \infty \cdot 0, \qquad CS \cdot DR = \infty \cdot 0, \qquad RT \cdot AR = \infty \cdot 0;$$
$$DA/DR = 0/0, \qquad AR/DR = 0/0, \qquad DR/DA = 0/0.$$

Again it appears from the obvious geometrical relations that

$$AS - CS = (CA)^2/(AS + CS), \qquad RS/ST = AR/OA, \qquad CS \cdot DA = DR \cdot CA,$$
$$DA/DR = AR/OR,$$

the right member of each of which approaches 0 as a limit;

$$OT - RT = OR, \quad ST/RT = OA/OR, \quad CS \cdot DR = CA \cdot OD, \quad AR/DR = OA/OR,$$

and the right members of these approach the limits $2a$, 1, $2a^2$, 1, respectively;

finally

$$OT - PT = OP, \quad RT/RS = CS/CA, \quad RT \cdot AR = RS \cdot OR, \quad DR/DA = CS/CA,$$

the right member of each of which becomes infinite.

We have therefore as the limits of the ratios involved

$$\begin{array}{lll} AS - CS \to 0, & OT - RT \to 2a, & OT - PT \to \infty; \\ RS/ST \to 0, & ST/RT \to 1, & RT/RS \to \infty; \\ CS \cdot DA \to 0, & CS \cdot DR \to 2a^2, & RT \cdot AR \to \infty; \\ DA/DR \to 0, & AR/DR \to 1, & DR/DA \to \infty. \end{array}$$

RECENT PUBLICATIONS.

REVIEWS.

Elementary Calculus. By W. F. Osgood. New York, The Macmillan Company, 1921. 12mo. 10 + 224 pages. Price $2.40.

Preface: "The object of this book is to present the elements of the Differential Calculus in a form easily accessible for the undergraduate. It is possible, from the very beginning, to illustrate the ideas and methods of the Calculus by means of applications to physics and geometry, which the student can readily grasp, and which will seem to him of interest and value. To do this, the stress in the illustrative examples worked in the text must be laid first of all on the thought which underlies the method of solution, in distinction from the exposition of a process, reduced in the worst teaching to rules, whereby the answer can be obtained. The treatment of maxima and minima, Chapter III, §§ 2, 3, and curve tracing, Chapter III, § 5, and Chapter VII, § 10, will serve to show what is here meant.

"It is, however, also essential that the student receive thorough training in the formal processes and the technique of the Calculus, and this side has been treated with care and completeness. Note, for example, the differentiation of composite functions in Chapter II, § 8, and the exposition of the use of differentials in differentiating in Chapter IV, §§ 4, 5.

"An important application of the graphical methods, with which the Calculus is so intimately associated, is that of solving approximately numerical equations which do not come under the standard rules of algebra and trigonometry. Hitherto, however, little attempt has been made to present this subject, simple as it is, in any systematic and elementary manner. In Chapter VII the common methods in use by physicists and others who apply the Calculus are set forth and illustrated by simple examples.

"The book might have included a brief treatment of curvature and evolutes, and the cycloid. But probably most teachers of the Calculus will prefer to take up integration next, and so the closing chapter is devoted to the last of the elementary functions, the inverse trigonometric functions, with special reference to their one great application in the elements of mathematics, namely, their application to integration.

"The book is so written that it can be adapted, if desired, to an abridged course, in which, after the fundamentals of the first three chapters have been covered, any of the remaining topics can be treated briefly, and thus a wide scope in subject matter is possible, even when the time is short."

Contents—Chapter I: Introduction, 1–12; II: Differentiation of algebraic functions. General theorems, 13–45; III: Applications, 46–80; IV: Infinitesimals and differentials, 81–104; V: Trigonometric functions, 105–145; VI: Logarithms and exponentials, 146–165; VII: Applications, 166–205; VIII: The inverse trigonometric functions, 206–224.

The Dynamics of the Airplane. (Mathematical Monographs, No. 21). By KENNETH P. WILLIAMS, Associate professor of mathematics, Indiana University. New York, John Wiley & Sons, 1921. 8vo. 8 + 138 pp. Price $2.50.

Preface: "It was the good fortune of the author to attend the University of Paris during the spring semester of 1919. One of the special courses which the French authorities, with their characteristic hospitality, arranged for the large number of students from the American army, was a course in aërodynamics, given by Professor Marchis. The comprehensive knowledge that Professor Marchis possessed of all branches of the new science of aëronautics, the inestimable value of his advice to the French Republic during the war, the interest he took in his rather unusual class, could not fail to be an inspiration.

"This book is an outgrowth of those parts of Professor Marchis' lectures that were of particular interest to the author. It is in no sense a complete treatise on aviation. Questions of design and construction are passed over with bare mention. The book is intended for students of mathematics and physics who are attracted by the dynamical aspect of aviation. The problems presented by the motion of an airplane are novel and fascinating. They vary from the most pleasing simplicity to the most stimulating difficulty. The question of stability, particularly, exhibits at the same time the elegance and the power of analysis, and shows the adaptability of some of the general developments in dynamics. The field is assuredly a fruitful one of study, and increasing demands will be put upon the mathematician as the science of aviation continues its rapid development. The mathematician can well own a sense of pride that he had already at hand, in the developments inaugurated by Euler and Routh, a means of dealing accurately with the question of stability, that plays so fundamental a rôle in the science of flying.

"The treatment in the text is for the most part elementary. The last chapter alone demands of the student familiarity with more advanced dynamical methods. In the treatment of descent a slight digression is made to consider in part the nature of the solution of a system of two differential equations. This was done in order not to completely evade what seems a problem of considerable difficulty. It might seem that a treatment of the propeller should not find a place in a book with the purpose of this one. No student of mathematics, however, could fail to own a curiosity as to a propeller's action, and it is hoped the discussion, while not complete, will at least serve as a sufficient introduction."

Contents—Chapter I: The plane and cambered surface, 1–17; II: Straight horizontal flight, 18–30; III: Descent and ascent, 31–51; IV: Circular flight, 52–68; V: The propeller, 69–85; VI: Performance, 86–93; VII: Stability and controllability, 94–106; VIII: Stability (continued), 107–129; Appendix, 131–136; Index, 137–138.

NOTES.

It is announced that John Wallis's *Arithmetica Infinitorum*, translated and edited by J. M. Child, is soon to be published by The Open Court Company, London—The second edition of Hobson's *The Theory of Functions of a Real Variable* is to be in two large volumes. The first volume appeared early in this year (Cambridge University Press; 16 + 671 pages; price 45 shillings).

At the meeting of the Academy of Sciences of the Institute of France last December, several sums of money from the Loutreuil Foundation were voted to assist scientific publications. Among them was 3000 francs to T. Lemoyne and H. Brocard "for the publication of the second and third volumes of their work *Courbes géométriques remarquables (courbes spéciales) planes et gauches*" (*1920*, 130).

The MONTHLY draws the attention of mathematicians to the excessive charges for publications of the Cambridge University Press made by the present American agent, The Macmillan Company. A single illustration will suffice.

The third edition of Whittaker and Watson's *A Course of Modern Analysis* was published at 40 shillings (*1921*, 31); the American agent's price is $12.50. Hence by ordering from a London bookseller, and paying the duty, a saving of at least $3.00 on the purchase of this single volume could be effected.—There is no duty for books ordered for college and public libraries. The American Branch of the Oxford University Press appears to count more definitely on the ignorance of purchasers in the United States. To illustrate: H. Hilton's *Plane Algebraic Curves*, 1920, was published at 28 shillings (about $5.60 at the present rate of exchange); the price of the American Branch is $12.60!

The second and concluding number of *Mathematische Annalen*, volume 80, is to be a Generalregister of volumes 51–80. The Generalregister, volumes 1–50 (11 + 202 pp.), was published in 1898.

Julius Springer, of Berlin, published early in 1921, the first volume (price 186 marks, "Zuschlagfrei," 12 + 612 pages + portrait) of Felix Klein's *Gesammelte Mathematische Abhandlungen*, "herausgegeben von R. FRICKE und A. OSTROWSKI (von F. Klein mit ergänzenden Zusätzen versehen)." The volume contains: "Liniengeometrie, Grundlegung der Geometrie zum Erlanger Programm."

In *Jahresbericht der deutschen Mathematiker-Vereinigung*, volume 29, nos. 1–6, September, 1920, we find (pp. 28–40) "Ein Beitrag zur Lebensbeschreibung von L. Fuchs" by L. SCHLESINGER, who has for years been collecting material for a life of FUCHS. Schlesinger refers to LEO KÖNIGSBERGER'S book, *Mein Leben* (Heidelberg, 1919), where there are various "assertions and judgments concerning L. Fuchs which do not in any wise correspond to the picture which I bear in my heart of my unforgettable teacher, and fatherly friend." In support of his contentions Schlesinger publishes several letters written by Weierstrass in 1875.

We have already referred (*1921*, 32) to the first of a series of memoirs by G. H. HARDY and J. E. LITTLEWOOD, on "Some problems of 'partitio numerorum'" published in the Göttingen *Nachrichten*, 1920, pp. 33–54. The second memoir was published January 31, 1921, in *Mathematische Zeitschrift*, volume 9, nos. 1–2, pp. 14–27, and deals with the "Proof that every large number is the sum of at most 21 biquadrates." The same issue of the *Zeitschrift* contains an article on "Congruence properties of partitions" by S. Ramanujan (see *1920*, 338). This is extracted by G. H. Hardy from one of the author's manuscripts which "contains a large number of further results. It is very incomplete, and will require very careful editing before it can be published in full. I have taken from it," Hardy writes, "the three simplest and most striking results, as a short but characteristic example of the work of a man who was beyond question one of the most remarkable mathematicians of his time."

Nature for February 17, 1921, was a special number on relativity, and contained the following articles, pages 782–811: "A brief outline of the development

of the theory of relativity" by A. EINSTEIN; "Relativity: the growth of an idea" by E. CUNNINGHAM; "Relativity and the eclipse observations of May, 1919" by F. DYSON; "Relativity and the motion of Mercury's perihelion" by A. C. D. CROMMELIN; "The displacement of solar lines" by C. E. ST. JOHN; "Non-Euclidean geometries" by G. B. MATHEWS; "The general physical theory of relativity" by J. H. JEANS; "The Michelson-Morley experiment and the dimensions of moving bodies" by H. A. LORENTZ; "The geometrisation of physics, and its supposed basis on the Michelson-Morley experiment" by O. LODGE; "Electricity and gravitation" by H. WEYL; "The relativity of time" by A. S. EDDINGTON; "Theory and experiment in relativity" by N. CAMPBELL; "The relation between Geometry and Einstein's theory of gravitation" by DOROTHY WRINCH and H. JEFFREYS; "The metaphysical aspects of relativity" by H. W. CARR. The "Bibliography of relativity" (pages 811–813) lists (1) 83 books and pamphlets (among the names of authors are: F. S. WOODS, E. V. HUNTINGTON, R. D. CARMICHAEL, L. SILBERSTEIN, R. C. TOLMAN and E. E. SLOSSON); and (2) the articles on the subject, which have appeared in *Nature*. These were mainly selected from a bibliography of about 650 titles for the period 1886–1920, compiled by H. F. Morley, director of the *International Catalogue of Scientific Literature*.—This special number of *Nature* was in great demand and was entirely out of print on March 17.

A new part of the great *New English Dictionary on Historical Principles*, Visor-Vywer, was issued from the Clarendon Press in March, 1920 (see *1919*, 256–257, *1920*, 128–129). The words in the section of special interest to the mathematician are: viss, volume and vortex (and their derived words and compounds), vorty, vulcan, vulgar fractions, and vulpecula—Viss is a weight used in southern India and Burmah equal to about three and one half pounds; it has appeared in English publications since 1588—Of the article on volume, paragraphs 8–9 give illustrations of the use of the word since 1792 as "the bulk, size, or dimensions of a thing," "the mass or solid body of something," "the amount or quantity of something," and "without article: bulk, mass, dimensions." Volumenometer, volumometer, volumenometry and volumetry are illustrations of derived words—Vortex has been used since its introduction in older theories of the universe, especially that of Descartes. Derived words are vorticose, vorticular and vortiginous—Vorty, a "south-western dialect form for forty," occurs in seventeenth century literature—Vulcan a hypothetical planet supposed to have its orbit between the sun and mercury—The earliest quotation for the term "vulgar fractions" is in Jeake's *Arithmetic*, 1696. The obsolete expression "vulgar arithmetic" was employed in the seventeenth and eighteenth centuries —Vulpecula is a small northern constellation lying between Hercules and Pegasus.

Among universities of America which have recently published complete lists of those on whom the degree of doctor of philosophy, with mathematics as a major, has been conferred are: (*a*) The University of Chicago, (*b*) Harvard University, and (*c*) The Johns Hopkins University.

(*a*) This list is in *Circular of the Departments of Mathematics, Astronomy and Astrophysics, Physics, Chemistry, 1920*. There have been 85 mathematical doctors, 1896–1919, L. E. DICKSON and J. I. HUTCHINSON being the first. With mathematical astronomy as a major, there were 11 more, 1900–1915. No indication is given as to whether or not the dissertations have been published, although it is known that this occurred in most cases.

(*b*) This list is to be found in *Official Register of Harvard University, Division of Mathematics, 1920–1921*. There were 47 doctors in mathematics, 1873–1919, the first being W. E. BYERLY. Indications are given of the place of publication of 32 of the dissertations.

(*c*) *Doctors' Dissertations 1878–1919* is the title of *The Johns Hopkins University Circular*, new series, 1920, no. 1. In mathematics there were 69 doctors, 1878–1918, the first being THOMAS CRAIG; in astronomy there were 5, 1891–1898. Indications are given regarding the publication of all of the dissertations except for 14 in mathematics and 2 in astronomy. Of these, 24 were published in the *American Journal of Mathematics*.

Since 1896, according to the above mentioned lists for mathematics, Harvard has had 41 doctors and Johns Hopkins, 46; so that during the period 1896–1919 the University of Chicago has conferred almost as many of such doctorates as Harvard and Johns Hopkins together.

The following extracts are from John Dewey's *Reconstruction in Philosophy* (New York, Holt, 1920).

"Mathematics is often cited as an example of purely normative thinking dependent upon *a priori* canons and supra-empirical material. But it is hard to see how the student who approaches the matter historically can avoid the conclusion that the status of mathematics is as empirical as that of metallurgy. Men began with counting and measuring things just as they began with pounding and burning them. One thing, as common speech profoundly has it, led to another. Certain ways were successful—not merely in the immediately practical sense, but in the sense of being interesting, of arousing attention, of exciting attempts at improvement. The present-day mathematical logician may present the structure of mathematics as if it had sprung all at once from the brain of a Zeus whose anatomy is that of pure logic. But, nevertheless, this very structure is a product of long historic growth, in which all kinds of experiments have been tried, in which some men have struck out in this direction and some in that, and in which some exercises and operations have resulted in confusion and others in triumphant clarifications and fruitful growths; a history in which matter and methods have been constantly selected and worked over on the basis of empirical success and failure.

"The structure of alleged normative *a priori* mathematics is in truth the crowned result of ages of toilsome experience. The metallurgist who should write on the most highly developed method of dealing with ores would not, in truth, proceed any differently. He too selects, refines, and organizes the methods which in the past have been found to yield the maximum of achievement. Logic is a matter of profound human importance precisely because it is empirically founded and experimentally applied. So considered, the problem of logical theory is none other than the problem of the possibility of the development and employment of intelligent method in inquiries concerned with deliberate reconstruction of experience. And it is only saying again in more specific form what has been said in general form to add that while such a logic has been developed in respect to mathematics and physical science, intelligent method, logic, is still far to seek in moral and political affairs" (pages 137–138). "Such a deductive science as mathematics represents the perfecting of method. That a method to those concerned with it should present itself as an end on its own account is no more surprising than that there should be a distinct business for making any tool. Rarely are those who invent and perfect a tool those who employ it. There is,

indeed, one marked difference between the physical and the intellectual instrumentality. The development of the latter runs far beyond any immediately visible use. The artistic interest in perfecting the method by itself is strong—as the utensils of civilization may themselves become works of finest art. But from the practical standpoint this difference shows that the advantage as an instrumentality is on the side of the intellectual tool. Just because it is not formed with a special application in mind, because it is a highly generalized tool, it is the more flexible in adaptation to unforeseen uses. It can be employed in dealing with problems that were not anticipated. The mind is prepared in advance for all sorts of intellectual emergencies, and when the new problem occurs it does not have to wait till it can get a special instrument ready" (page 149).

ARTICLES IN CURRENT PERIODICALS.

AMERICAN OXONIAN, Concord, N. H., volume 8, no. 1, January, 1921: "The record of the American Rhodes Scholars" by R. W. Burgess, 1–36 [Of the 351 American Rhodes Scholars at Oxford, classes matriculating 1904–1914, 32.7 percent studied law; 17.1 percent modern history and economics; 16.8 percent humanities, including the classics, philosophy, and anthropology; and 6 percent mathematics, physics, chemistry and engineering].

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 27, no. 4, January, 1921: "The October meeting of the San Francisco Section" by B. A. Bernstein, 149–153; "An image in four-dimensional lattice space of the theory of the elliptic theta functions" by E. T. Bell, 153–160; "Note on the median of a set of numbers" by D. Jackson, 160–164; "Note on closure of orthogonal sets" by O. D. Kellogg, 165–169; "The mathematical work of Thomas Jan Stieltjes" [review of *Oeuvres complètes de Thomas Jan Stieltjes* (2 volumes, Groningen, 1914–1918)] by R. D. Carmichael, 170–178; Reviews by D. E. Smith of *Opere di Evangelista Torricelli* (edited by G. Loria and G. Vassura, 2 volumes, Faenza, 1919) and of W. W. R. Ball's *An Introduction to String Figures* (Cambridge, 1920), 178–181; Review by C. N. Moore of A. R. Forsyth's *Solutions of the Examples in a Treatise on Differential Equations* (London, 1918), 181–182; Reviews by E. B. Wilson of A. S. Eddington's *Space, Time, and Gravitation; an Outline of the General Relativity Theory* (Cambridge, 1920) and of R. W. Wood's *Researches in Physical Optics* (New York, 1919), 182–186; Notes, 187–193; New publications, 194–196.

EDUCATIONAL ADMINISTRATION AND SUPERVISION, volume 6, no. 9, December, 1920: "The scoring of geometry test W" by J. H. Minnick, 509–511.

HIBBERT JOURNAL, volume 19, January, 1921: Review by C. D. Broad of A. N. Whitehead's *The Concept of Nature* (Cambridge, 1920), 360–366 [Last paragraph: "The thanks rendered in the preface by Professor Whitehead to the Cambridge University Press officials seem to me excessive. No doubt their hearts are in the right place, but they have passed at least six bad mistakes. On p. 51, l. 4, for *sight* read *touch;* p. 86, l. 8, for *external, eternal;* p. 148, l. 4, for *agree, argue;* p. 155, l. 17, for *sense-object, perceptual object;* p. 180, l. 4, for *universely* (a pleasant conceit!), *inversely;* and on p. 188, l. 9, for *by* read *from.* In conclusion, I must say that anyone who has read *Principles of Natural Knowledge* will find his understanding of that book much improved by reading *The Concept of Nature;* and that anyone who has read neither should go at once to his (or her) bookseller and order both."]

JOURNAL OF PHILOSOPHY, volume 28, no. 2, January 20, 1921: "Eddington on Einstein" by E. E. Slosson, 48–51 [Last paragraph: "Some mathematicians and physicists have manifested impatience at the impertinent curiosity of the public and declare that Einstein's theory concerns only themselves, and whatever they may decide to do with it can have no possible effect upon anybody's religion, philosophy or view of life. But the public knows better. And Professor Eddington agrees with the majority on this question. Galileo, Newton and Darwin were specialists, speculating in fields remote from common life, yet they have revolutionized the thought and altered the conduct of the world. Einstein's theory is even more fundamental and uncontentional and if it is verified by experiment or generally adopted as a working hypothesis it will be found in the course of time to have a profound influence upon the minds of men outside of the realm of science."]

JOURNAL OF THE UNITED STATES ARTILLERY, volume 51, December, 1919: "Charts for the calculation of the effect of small changes in the elements of fire" by P. L. Alger, 585–603—Volume 53, August, 1920: "Two misconceptions" by R. S. Hoar, 179–181—October: "On weighting factor curves for flat fire" by J. F. Ritt, 404–410—December: "Wind weighting factors" by J. J. Johnson, 578–587; "Mirror and window position finders" by W. C. Graustein, 588–610.

MATHEMATICAL GAZETTE, volume 10, January, 1921: "Unicursal plane curves" by G. B. Mathews, 193–194; "The lighter side of mathematics" by C. A. Stewart, 195–200 ["In our profession we often come into contact with those who do not understand Pure Mathematics. Some of these are respectful as if entering a shrine; but others, of the baser sort, are contemptuous. My object in this paper is to consider the position of the mathematician in two of his possible moods—during his moments of leisure and during his moments of reflection]; "Gleanings far and near," 200, 219; "The tracing of conics" by E. H. Neville, 201–203; "An odd method for determining the year of birth" by G. A. Miller, 208–209; "Lagrange's tribute to Maclaurin" by C. Tweedie, 209; "Note on the differentiation of sin x, and on the limit of $(\sin x)/x$ as x tends to zero" by C. H. Hardingham, 212–215. [G. A. Miller's note[1]: "In the second edition of Cajori's *History of Mathematics*, 1919, page 330, there appears an interesting biographical sketch of Augustus De Morgan, the second sentence of which is as follows: 'For the determination of the year of his birth (assumed to be in the nineteenth century) he proposed the conundrum, "I was x years of age in the year x^2."' It may be of interest to note that after the year 1936 the conditions here given are insufficient to determine the year of birth of DeMorgan, since

$$1806 + 43 = (43)^2 \text{ and } 1892 + 44 = (44)^2.$$

"Perhaps the interest in the given remark, which is said[2] to have been made by DeMorgan, is enhanced by the observation that in every later century there is no more than one year, such that by adding an integer x to it there results a sum which is equal to x^2. For instance, if a man born in the twentieth century will be x years old in the year x^2, he must be born in 1980, since

$$1980 + 45 = (45)^2.$$

"For the twenty-first and twenty-second centuries the corresponding years are 2070 and 2162 respectively, since

$$2070 + 46 = (46)^2 \text{ and } 2162 + 47 = (47)^2.$$

"The thirty-third century is the first century which does not contain a year such that if you add x to it you obtain the year x^2, and the fifteenth century is the next to the nineteenth in which there are two such years; viz: 1406 and 1482, since

$$1406 + 38 = (38)^2 \text{ and } 1482 + 39 = (39)^2.$$

"These very elementary observations tend to show that if DeMorgan made the said remark it does not exhibit his usual thoughtfulness and accuracy."] [C. Tweedie's note: "In the recent work by Professor Cajori on the *History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse*, the author preludes his remarks upon Colin Maclaurin's *Treatise of Fluxions* with the following statement:

"'Maclaurin's book on fluxions has been considered the ablest and most rigorous text of the eighteenth century. It was pronounced by Lagrange, 'le chef d'oeuvre de géométrie qu'on peut comparer à tout ce qu'Archimède nous a laissé de plus beau et de plus ingénieux.''

"And there is a footnote '*Mém. de L'Acad. de Berlin*, 1773; quoted in the art. Maclaurin in Sidney Lee's *Dict. of National Biography*.'

"The statement by Lagrange occurs in his memoir, 'Sur l'attraction des Sphéroides Elliptiques' (*Mém. de L'Acad. de Berlin*, 1773, p. 121).

"Here is the passage: 'M. Maclaurin qui a le premier résolu ce problème dans son excellent Pièce sur le flux et reflux de la mer, couronnée par l'Académie des Sciences de Paris en 1740, a suivi une méthode purement géométrique, et fondée uniquement sur quelques propriétés de l'ellipse et des sphéroides elliptiques: et il faut avouer que cette partie de l'ouvrage de M. Maclaurin est un chef-d'oeuvre de Géométrie qu'on peut comparer à tout ce qu'Archimède nous a laissé de plus beau et de plus ingénieux.'

"Clearly this has nothing whatever to do with the *Treatise of Fluxions* of Maclaurin, and it is most unfortunate that such a sublime appreciation of the work of one great mathematician by another should be so cruelly misapplied by two such eminent authorities. There may have been confusion in the mind of the earlier writer for the *D.N.B.* between *flux* and *fluxion*, but such an excuse cannot be tendered for a scholar of Professor Cajori's standing.

[1] The gist of this note appeared in *School and Society*, August 7, 1920. Cf. this MONTHLY, *1920*, 476.

[2] "[v. *Budget of Paradoxes*, p. 332; 2nd ed., p. 124, vol. ii. All that DeMorgan says is, 'I was x years old in A.D. x^2'; not 4 in A.D. 16, nor 5 in A.D. 25, but still in one case under that law.]"

"The passage from Lagrange is correctly quoted in the *Life of Maclaurin* published in the *Mathematical Gazette*, October 1916, and is to be found also in Chasles' *Aperçu Historique*.

"Maclaurin was, of course, 'the creator of the theory of the attraction of ellipsoids.'"]

MATHEMATICS TEACHER, volume 14, no. 1, January, 1921: "The National Council of Teachers of Mathematics" by C. M. Austin, 1–4; "Progress of the National Committee on Mathematical Requirements" by J. W. Young, 5–15; "Junior high school mathematics: A discussion of the National Committee Report" by C. B. Walsh, E. R. Breslich, W. Betz, Marie Gugle, R. Schorling, 16–41; "First lessons in demonstrative geometry" by M. J. Newell and G. A. Harper, 42–45; Editorials, News and Notes, Questions and Answers, Book Reviews, 46–54.

MESSENGER OF MATHEMATICS, volume 50, nos. 2 and 3, June and July, 1920: "4-tic & 3-bic residuacity-tables" (continued) by A. Cunningham and T. Gosset, 17–30; "On plane curves of degree n with tangents of n-point contact (second paper)" by H. Hilton, 31–40; "On Laplace's theorem of simultaneous errors" by L. V. Meadowcroft, 40–48—No. 4, August: "Four-vector algebra and analysis (part II)" by C. E. Weatherburn, 49–61; "On a Diophantine problem (third paper)" by H. Holden, 62–64.

NATURE, volume 106, December 23, 1920: Review of E. T. Whittaker and G. N. Watson's *A Course of Modern Analysis* (3d edition, Cambridge, 1920), 531—December 30: "Mathematics in secondary education" [in the United States], 583–584—January 6, 1921: "Nomography" [review of S. Brodetsky's *A First Course in Nomography* (London, 1920)], 593–594—January 13: "The Mathematical Association" by C. G., 644–645—January 20: "General dynamics" by A. Gray, 655–656 [Review of Lamb's *Higher Mechanics* (Cambridge, 1920)]; "The history of determinants" by G. B. M., 658 [Review of Muir's *The Theory of Determinants*, vol. 3 (London, 1920)]; "The mechanics of solidity" by J. Innes, 662–663; "Measurements of the angular diameters of stars" by A. C. D. C., 676–677; "The late Srinivasa Ramanujan" by E. H. Neville, 661–662 [Last two paragraphs: "Ramanujan walked stiffly, with head erect, and his arms, unless he was talking, held clear of his body, with hands open and palms downward. In conversation he became animated, and gesticulated vividly with his slender fingers. He had a fund of stories, and such was his enjoyment in telling a joke that often his words struggled incomprehensible through the laughter with which he anticipated the climax of a narrative. He had serious interests outside mathematics; he was always ready to discuss whatever in philosophy or politics had last caught his attention, and Indians speak with admiration of a mysticism of which his English friends understood little.

"Perfect in manners, simple in manner, resigned in trouble and unspoilt by renown, grateful to a fault and devoted beyond measure to his friends, Ramanujan was a lovable man as well as a great mathematician. By his death I have suffered a personal loss, but I do not feel that his coming to England is to be regretted even for his own sake. Prof. Hardy speaks of disaster because of the hopes he entertained. If he pictures Ramanujan as he might have been throughout a long life, tormented by a lonely genius, unable to establish effective contact with any mathematicians of his own class, wasted in the study of problems elsewhere solved, Prof. Hardy must agree that the tragedy averted was the greater. Shortly before he left England, at a time of great depression, Ramanujan told me that he never doubted that he did well to come, and I believe that he would have chosen as he did in Madras in 1914 even had he known that the choice was the choice of Achilles."]

NOUVELLES ANNALES DE MATHÉMATIQUES, volume 79, December, 1920: "Charles-Ange Laisant (1841–1920)" by R. Bricard, 449–454; "Étude des surfaces de translation de Sophus Lie" (suite et fin) by B. Gambier, 454–479; "Sur certaines relations qui existent entre l'épicycloide et l'hypocycloide à trois rebroussements" by J. A. Moren, 479–484; "Considérations sur le frottement de glissement" by E. Delassus, 485–496; "Note géométrique sur une généralisation du théorème de composition des vitesses et le théorème de Coriolis" by L. Pomey, 496–501; "Table des matières par ordre méthodique," 503–507.

PHILOSOPHICAL REVIEW, volume 30, January, 1921: "Dr. Whitehead's theory of events" by D. S. Robinson, 41–56 [First paragraph: "Dr. Whitehead has rightly said: 'It is a safe rule to apply that, when a mathematical or philosophical author writes with a misty profundity, he is talking nonsense.'[1] Now much of his own writing is assuredly impervious to this criticism, being crystal-clear as well as genuinely profound. But does not his singular and noteworthy theory of events, as recently expounded in his *Enquiry concerning the Principles of Natural Knowledge*,

[1] "*Introduction to Mathematics*, p. 227."

supply suitable material for the application of this 'safe rule'? If, as Dr. Whitehead claims, events are the ultimate facts of nature and the ultimate data of science, it is manifestly important that philosophers should have accurate and clear knowledge of what an event is. But a careful study of his account has convinced me that it is needlessly abstruse and nebulous, indeed, filled with what may well be called misty profundity. The attempt to substantiate this contention involves a somewhat minute examination of that part of his exposition setting forth his conception of an event."]

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, volume 6, no. 10, October, 1920: "Motion on a surface for any positional field of force" by J. Lipka, 621–624.

REVUE GÉNÉRALE DES SCIENCES PURES ET APPLIQUÉES, volume 31, December 15, 1920: Review by C. Maillard of Karpinski, Benedict and Calhoun's *Unified Mathematics* (Boston, 1920), 765.

REVUE PHILOSOPHIQUE DE LA FRANCE ET DE L'ETRANGER, volume 89, March–April, 1920: "Les idées de temps, de durée et d'éternité dans Descartes" by J. Vigier, 196–223—May–June: "Les idées de temps etc." (suite) by J. Vigier, 312–348—Volume 90, November–December: "Un savant français: Henri Poincaré" by A. Denjoy, 321–350 [First paragraph: "Quand Poincaré mourut en 1912 à l'âge de cinquante-neuf ans, il laissait une oeuvre immense, dont le mérite recueillait le suffrage unanime des représentants de toutes les catégories du savoir qui empruntent aux mathématiques une aide indispensable. Le nombre de ses notes, articles, mémoires parus dans des publications périodiques réservées à la science, approchait 500. De grands ouvrages de mécanique céleste, quatorze volumes de Physique mathématique, et enfin,—originalité lui conférant une place unique parmi tous les grands mathématiciens des deux derniers siècles,—trois volumes philosophique s'ajoutaient, pour en accroître notablement l'importance, aux pièces fragmentaires déjà signalées. Et l'on ne doit point s'imaginer que cette énorme masse de text sortie de la plume de notre savant, se fût artificiellement enflée par la prolixité de son auteur. Les lectures de Poincaré qui porteraient de ce point de vue un jugement sur son oeuvre, ne songeraient pas précisement à lui adresser ce reproche-là. L'expression est tout juste suffisamment étendue pour déterminer sans aucune équivoque le sens de la pensée. Selon la bonne règle française, il ne demeure pas un seul membre de phrase dont la disparition trouverait suppléance dans le texte restant. Cette production formidable d'écrits représente uniquement une somme incompressible d'idées dont l'éclusion ne pouvait se dérouler que dans l'un des cerveaux les plus fertiles de tous les temps."]; "Descartes et Harve" (à suivre) by E. Gilson, 432–458.

SCHOOL SCIENCE AND MATHEMATICS, volume 21, no. 1, January, 1921: "Minimum high school mathematics" by F. Cajori, 25–28 [First paragraph: "A pupil permitted to graduate from a high school without any mathematics is in danger of remaining the unenvied occupant of a blind alley. The attempt to reduce the minimum high school course in mathematics to one year is unwise, because thus far experience shows that one year is too short a time for a pupil to acquire a knowledge of elementary mathematics that can be applied in practical life. Two years would seem to be a better minimum."]; "The first month of geometry" by J. A. Nyberg, 29–36; "High school and college mathematics" by T. E. Mason, 37–44; Problem department, 83–86—February: "The teaching of graphs" by J. A. Nyberg, 144–149; "Tests of mathematical ability and their prognostic values. A discussion of the Rogers tests" by L. E. Mensenkamp, 150–162; Problems and solutions, 173–177.

SCIENCE, new series, volume 53, January 14, 1921: "Romancing in science" by D. W. Horn, 44 [With reference to F. Cajori's communication listed in this MONTHLY, *1921*, 138 (see also contents of *Science*, *1921*, 35)]—January 28: "Musical notation" by T. P. Hall, 91–92 [First paragraph: "To the Editor of *Science*: In the September number of *The Scientific Monthly* Professor E. V. Huntington describes a new way of writing music [cf. this MONTHLY, *1921*, 35], which for simplicity and clearness can hardly be surpassed. It consists in using the ordinary staff for the twelve notes of the tempered chromatic scale, instead of (as now done) for the seven notes of the diatonic scale. This new 'normalized' notation does away with all sharps and flats. Since there are just twelve lines and spaces (including the added line below) in each staff, each letter will have always the same position on the staff, whether soprano, alto, tenor or bass. It is hoped that teachers will take advantage of the normalized notation to smooth out the road for beginners, particularly in the grade schools."]; "Star-time observations with an engineer's Y-level" by W. J. Fisher, 94–95—February 4: "The biographical directory of American men of science" by J. McK. Cattell, 118; Review by L. S. Marks of F. Bedell's *The Airplane* (New York, 1920), 119–120; "The Washington conference on the history of science" by L. Thorndike, 122 [First

paragraph: "The conference upon the History of Science, initiated by the American Historical Association at its annual meeting a year ago in Cleveland, proved such a success that the program committee devoted another session to the subject this December at Washington. Simultaneously the History of Science Section, which has recently been formed under the auspices of the American Association for the Advancement of Science, was meeting in Chicago, thus demonstrating the widespread interest in this promising field. This widespread interest was further evidenced at Washington by the variety of learned occupations represented by the speakers, who included, in addition to professors of science and history, a librarian, a college president, and the head of an institution for research."]—February 11: "Reply to Professor Horn" [see above] by F. Cajori, 139.

SCIENCE PROGRESS, volume 15, January, 1921: "The $\sqrt{-1}$: a protest" by "Amateur," 456–457 [First few sentences: "Probably modern mathematics differs from past mathematics chiefly in the stress which is now laid upon 'Complex Numbers.' When algebra was first invented numbers were conceived to be signless; then gradually the so-called negative numbers were introduced; then mathematicians went on to separate rational from irrational numbers; and now their pupils are obliged to twist their brains by the consideration of complex numbers. Every book one reads nowadays commences with a series of paragraphs on these different kinds of numbers, and the reader is often obliged to generalize the simplest functions in terms of the last mentioned. Is there really any advantage in all this? And, though I am only an amateur, I should like to maintain that there is no such advantage, and, moreover, that complex numbers do not exist at all—though I am aware that such a statement will expose me to adverse or even contemptuous criticism. To begin with, it may be of course even doubted whether there are such things as negative numbers—and this doubt has been frequently expressed by the greatest experts. Negativeness is not a property of number itself but merely an expression of the fact that a number has been subjected to the inverse operation of addition."]; Review by R. R[oss] of Cajori's *History of the Conceptions of Limits and of Fluxions in Great Britain from Newton to Woodhouse* (London, 1919), 486–487.

SCIENTIFIC AMERICAN, volume 123, September 18, 1920: "New concepts of the past century. The change in outlook since classical days, which makes non-Euclidean geometry a possibility" by the Einstein Prize Essay Editor, 276, 286, 288.

SCIENTIFIC MONTHLY, volume 12, no. 1, January, 1921: "On the character of primitive human progress" by R. D. Carmichael, 53–61 [First and last paragraphs: "The most remarkable thing among natural processes is the unfolding of the intellect and moral nature of man. Since his emergence from the animal state he has possessed powers comparable to those which he now manifests. Neither history nor speculation can reveal a period in his development when he was not making conquests evincing the same high order of intelligence as that which marks even his later career. In the earliest stages the individual man or the small group in a roving tribe had to approach the problems of life and environment without any effective tradition to guide or sympathetic collaboration with others to inspire. This called for a measure of independence unlike anything manifested by individuals today except in the labors of men of dominating genius. Among ruder peoples, in early times and at the present, the remarkable character of the discovery of truth is signalized by the acceptance of the new vision as something supernormal and sacred, akin to the activity of the gods and directly inspired by them. Though we have ceased to refer it to the supernatural, we ourselves understand it but little better. . . . We may take pleasure in such ancestors as our forefathers showed themselves to be even in the periods of savagery and barbarism through which we have rapidly sketched their development. They stood in the presence of phenomena whose nature was awe-inspiring to the creature that first inquired concerning their meaning. With no traditions to assist, with no previous conquests or discoveries of truth to start them out, with only a dumb and undeveloped sense of instinct of the destiny of man to light the way into a darkness of ignorance more profound perhaps than we can conceive today when so much of the push of the past has already been realized in our individual lives before we come to contemplate philosophically the nature of our environment and our relation to it, they began a career of development to which nothing else in our ken is to be compared."]; "The group-theory element of the history of mathematics" by G. A. Miller, 75–82 [First two paragraphs: "Few mathematical terms suggest such fundamental human cravings as the term group, and few have been more appropriately chosen. Just as human society has led to perplexities which increased with the advance of civilization, so the mathematical group-theory has given rise to problems which became more and more difficult with the advances in the development of mathematics. In both cases the primitive stages are comparatively simple and their history throws important light on the later developments.

"The history of the mathematical group-theory can be conveniently divided into three periods. The first of these extends from the beginning of mathematical history to about 1770 A.D., and may be called the *implicit period* since the group concept was then employed without being explicitly stated. The second, or *specialization period*, extends from about 1770 to about 1870. During this period the theory of substitution groups was founded as an autonomous science and the usefulness of this theory in the study of algebraic equations was emphasized. The third, or *generalization period*, extends from about 1870 to the present day, and is characterized by increased generalizations by abstraction and the explicit use of groups in each of the large domains of mathematics."]

TOHÔKU MATHEMATICAL JOURNAL, volume 18, nos. 3 and 4, December, 1920: "On continuous sets of points, II" (continued) by K. Yoneyama, 205–255; "On the separation theorem for the integrals of the differential equation of the third order" by K. Ôishi, 256–260; "On the distribution of electricity on two mutually influencing spheroidal conductors" by B. Datta, 261–267; "Sur quelques propriétés relatives à certains polygones inscrits à une circonférence" by V. Thébault, 268–272; "On some characteristic properties of curves and surfaces" by S. Nakajima, 272–287; "A note on the closed convex surface" by K. Ôishi, 288–290; "An application of the orthogonal trajectory to a problem of singular solutions of ordinary differential equations of the first order" by T. Matsumoto, 291–294; "On the integral $\int_0^\pi e^z \cos r\theta \sin q\theta, \cos q\theta \, d\theta$" by T. Takeuchi, 295–296; "Über lineare Gleichungen mit unendlich vielen Variabeln" by T. Kubota, 297–301; "Akiyuki Kemmochi and his 'Equal circles on the sphere'" [in Japanese] by T. Hayashi, 302–308; "Some formulas in the theory of interpolation of many independent variables" by S. Narumi, 309–321; "On the use of a table of double entry" by T. Hayashi, 322–326; "Shorter notices and reviews, Miscellaneous notes," 327–338.

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 22, no. 1, January, 1921: "Arithmetical paraphrases" by E. T. Bell, 1–30; "The construction of algebraic correspondences between two algebraic curves" by V. Snyder and F. R. Sharpe, 31–40; "Concerning certain equicontinuous systems of curves" by R. L. Moore, 41–55; "Fundamental systems of formal modular seminvariants of the binary cubic" by W. L. G. Williams, 56–79; "A property of two $(n+1)$-gons inscribed in a norm-curve in n-space" by H. S. White, 80–83; "Recurrent geodesics on a surface of negative curvature" by H. M. Morse, 84–100; "On the location of the roots of the jacobian of two binary forms, and of the derivative of a rational function" by J. L. Walsh, 101–116; "On functions of closest approximation" by D. Jackson, 117–128.

TRANSACTIONS OF THE ROYAL SOCIETY OF SOUTH AFRICA, Cape Town, volume 8, 1920, part 2: "Note on unimodular and other persymmetric determinants" by T. Muir, 95–101; "Note on certain determinant identities arrived at by H. v. Koch" by T. Muir, 101–105—Part 3: "Note on a sum of products which involves symmetrically the nth roots of 1" by T. Muir, 173–178; "Additional note on the resolvability of the minors of a compound determinant" by T. Muir, 229–233—Part 4: "Second note on the determinant of the sum of two circulant matrices" by T. Muir, 293–296.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. The editorial work of this department would be greatly facilitated if, on sending in problems, the proposers would also enclose their solutions—*when they have them.* If a problem proposed is not original the proposer is requested *invariably* to state the fact and to give an exact reference to the source.]

2896. Proposed by the late L. G. WELD.

A circle is inscribed within a triangle. In each of the three spandrels between this circle and the vertices another circle is described; in each of the three spandrels between these last circles

and the vertices another circle; and so on *ad infinitum*. Show that the ratio of the sum of the areas of all the circles to the area of the triangle is

$$\frac{\Sigma}{\Delta} = \frac{\pi}{4}\frac{\Delta}{S^2}\left[\frac{1}{\sin\frac{1}{2}A} + \frac{1}{\sin\frac{1}{2}B} + \frac{1}{\sin\frac{1}{2}C} - 2 + \sin\tfrac{1}{2}A + \sin\tfrac{1}{2}B + \sin\tfrac{1}{2}C\right].$$

2897. Proposed by PAUL CAPRON, U. S. Naval Academy.

Discuss the conditions under which the angles made by two circles on a sphere have the same measures as the distances between their poles.

2898. Proposed by J. W. CLAWSON, Ursinus College.

Four straight lines determine four triangles. It is well known that the circumcenters of these triangles lie on a circle and that the circumcircles intersect this circle in a point, called the Wallace point. It is also well known that the orthocenters of the four triangles lie on a straight line, which is perpendicular to the line on which lie the middle points of the three diagonals of the quadrilateral determined by the four given straight lines.[1]

Prove that the centroids of the four triangles lie on a parabola whose axis is parallel to the mid-diagonal line; and that the distance from the Wallace point to the mid-diagonal line is two-thirds of the distance from the Wallace point to the axis of the parabola.

2899. Proposed by NORMAN ANNING, University of Michigan.

A, B, C, and P are any four coplanar points. P describes a sextant about A when the line AP turns about A through $+60°$. Show that P moves in a closed curve when it describes sextants in succession *either* about $A, B, A, B, A, \cdots$ *or* about $A, B, C, A, B, C, \cdots$.

PROBLEMS—NOTES.

13. Professor C. N. MILLS, of Heidelberg University, Tiffin, Ohio, proposed the following problem: "A, D and C are telegraph poles at equal intervals by the side of a railroad; t_1 and t_2 are the tangents of the angles which AD and DC subtend at any point B on the road; t is the tangent of the angle which DB makes with DC. Show that $2/t = 1/t_1 - 1/t_2$." This relation may be written down at once from the familiar results[2] that $1/t = (\cot A - \cot C)/2$ where $A = \angle BAC$, $C = \angle BCA$; $1/t_1 = 2\cot B + \cot A$, where $B = \angle ABC$; and $1/t_2 = 2\cot B + \cot C$. If s_1, s_2 and s are the sines of the angles whose tangents are t_1, t_2 and t, respectively, it may be readily shown[3] that $s_1 = (a\sin B)/d$, $s_2 = (b\sin C)/d$, $s = (a\sin C)/d$, where a, b, c have the usual significance in connection with the triangle ABC, and $d = \sqrt{2a^2 + 2c^2 - b^2}$. ARC.

14. Professor I. A. BARNETT, of the University of Saskatchewan, proposed the problem: "Prove that

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15} = \left(\frac{1}{2}\right)^7."$$

This result is given as example 13 on page 319 of W. E. Johnson, *Treatise on Trigonometry*, London, 1889, and as example 5 on page 119 of E. W. Hobson, *A Treatise on Plane Trigonometry*, second edition, Cambridge, 1897. The solu-

[1] Steiner, *Annales de Mathématiques Pures et Appliquées*, April, 1828.—EDITOR.

[2] See, for example, J. Casey, *A Treatise on Plane Trigonometry*, Dublin, 1888, pp. 174–175; S. L. Loney, *Plane Trigonometry*, Cambridge, 1893, p. 246.

[3] S. L. Loney, *l.c.*, p. 240.

tion follows readily from the preceding theory where it is shown that

$$2^{\frac{1}{2}(n-1)} \sin\frac{\pi}{2n} \sin\frac{3\pi}{2n} \cdots \sin\frac{(n-2)\pi}{2n} = 1 \qquad \text{when } n \text{ is odd,}$$

and

$$2^{\frac{1}{2}(n-1)} \sin\frac{\pi}{2n} \sin\frac{3\pi}{2n} \cdots \sin\frac{(n-1)\pi}{2n} = 1 \qquad \text{when } n \text{ is even.}$$ ARC.

15. In the discussion of problem 2819 [*1920*, 134, 187–190] concerning the envelope of the system of circles inscribed in a triangle with a given base, AB, and a given vertical angle, C, it is remarked that, if we regard the variable triangle on one side of the base only, the locus of the centers of the system of circles is an arc of a circle through A and B, and whose center bisects the arc, on the opposite side of AB, of the circumscribing circle. This locus was noticed by John Turner, in *The Mathematician*, no. 5, 1749, p. 312, in the course of his answer to the problem he had proposed the previous year: One side of a triangle, together with the radii of its circumscribing and inscribed circles being given, to construct the triangle geometrically. The locus was also discussed by N. Fuss,[1] in 1794; by J. Bonnycastle,[2] in 1818; by D. Cresswell,[3] in 1819; and by Holleben and Gerwien,[4] in 1831, along with the corresponding locus for the escribed circle in C. These loci, and that for a second escribed circle, were called for by J. Luby[5] in 1833.

It is concerning the locus of the incenter that the philosopher Herbert Spencer wrote in his *Autobiography* (New York, 1904), volume 1, pp. 187–188:

"When seventeen I hit on a geometrical theorem of some interest. This remained with me in the form of an empirical truth; but during the latter part of my residence in Worcester, responding to a spur from my father, I made a demonstration of it; and, now that it had reached this developed form, it was published in *The Civil Engineer and Architect's Journal* for July, 1840. It is reproduced in Appendix B. I did not know, at the time, that this theorem belongs to that division of mathematics at one time included under the name 'Descriptive Geometry,' but known in more recent days as 'The Geometry of Position'—a division which includes many marvellous truths. Perhaps the most familiar of these is the truth that if to three unequal circles anywhere placed, three pairs of tangents be drawn, the points of intersection of the tangents fall in the same straight line—a truth which I never contemplate without being struck by its beauty at the same time that it excites feelings of wonder and of awe: the fact that apparently unrelated circles should in every case be held together by this plexus of relations, seeming so utterly incomprehensible. The property of a circle which is enunciated in my own theorem, has nothing like so marvellous an aspect, but is nevertheless sufficiently remarkable."

The article containing "my own theorem," [6] and a long-winded proof, is reproduced on pages 606–608 of his *Autobiography*, volume 1. The above quotation indicates that Spencer was innocent of knowledge of even the elementary ideas

[1] *Nova Acta Acad. Sc. Imp. Petrop.*, vol. 10 (1792), 1797, pp. 124–125.

[2] J. Bonnycastle, *Elements of Geometry*, sixth edition, London, 1818, pp. 310–311.

[3] D. Cresswell, *A Supplement to the Elements of Euclid*, Cambridge, 1819, pp. 256–257; second ed., 1825, pp. 227–228.

[4] H. v. Holleben and P. Gerwien, *Geometrische Analysis*, vol. 1, 1831, p. 43; vol. 2, 1832, p. 53.

[5] J. Luby, *The Elements of Geometry . . . Also a variety of problems and theorems . . . with analysis*, Dublin, 1833, third part, p. 57.

[6] Discussed, as we have seen, about a hundred years before. References have been given to only a few of the numerous earlier discussions of the theorem and its extensions.

of geometry.[1] This and other instances of such ignorance are set forth by J. S. Mackay in "Herbert Spencer and mathematics," *Proceedings of the Edinburgh Mathematical Society*, 1907, volume 25, pp. 95–106. ARC.

PROBLEMS—SOLUTIONS

2800 [1920, 31]. Proposed by A. M. HARDING, University of Arkansas.

If $x + y + z = xyz$, show that

$$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2} \cdot \frac{2y}{1-y^2} \cdot \frac{2z}{1-z^2}.$$

SOLUTION BY ARNOLD DRESDEN, University of Wisconsin.

Except for values of x and y for which $xy = 1$, the relation (1) $x + y + z = xyz$ is equivalent to $z = (x + y)/(xy - 1)$. Direct substitution of this value for z shows that the formula

$$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2} \cdot \frac{2y}{1-y^2} \cdot \frac{2z}{1-z^2} \tag{2}$$

holds for all values of x, y, except $x = \pm 1$; $y = \pm 1$; $xy = 1$, for which values one or the other of these equations loses meaning.

It is readily seen that an infinite number of other formulas will hold true as a consequence of (1), viz.,

$$\Sigma \tan (n \tan^{-1} x) = \Pi \tan (n \tan^{-1} x), \text{ for } n = 1, 2, \cdots, \tag{3}$$

the sum and product to be extended over x, y, and z. For since $x + y + z = xyz$, it follows that $-\tan^{-1} x - \tan^{-1} y = \tan^{-1} z$, for some determination of $\tan^{-1} z$; in virtue of this relation we have then $-n \tan^{-1} x - n \tan^{-1} y = n \tan^{-1} z$ and therefore:

$$-\tan (n \tan^{-1} z) = \tan (n \tan^{-1} x + n \tan^{-1} y) = \frac{\tan (n \tan^{-1} x) + \tan (n \tan^{-1} y)}{1 - \tan (n \tan^{-1} x) \cdot \tan (n \tan^{-1} y)};$$

i.e., $\Sigma \tan (n \tan^{-1} x) = \Pi \tan (n \tan^{-1} x)$.

It would be interesting to know the general solution of the functional equation:

$$\sum_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} f(x_i),$$

under the restriction $\Sigma x_i = \Pi x_i$.

Also solved by NORMAN ANNING, T. M. BLAKSLEE, S. M. BERG, J. A. BULLARD, H. N. CARLETON, E. S. HAMMOND, C. N. MILLS, E. J. OGLESBY, H. L. OLSON, J. L. RILEY, ARTHUR PELLETIER, H. S. UHLER, W. R. WARNE, and C. C. WYLIE.

2802 [1920, 31]. Proposed by WARREN WEAVER, California Institute of Technology.

Consider two circles, each of radius k, with centers at (0, 0) and (k', 0) respectively, where k' is less than k. Through the point (k', 0) draw a ray making an angle θ with the positive x-axis. Call the intersection of this line with the first circle A, and with the second circle B. Extend the line through the point (k', 0) in the opposite direction, and call the intersection of this extension with the first circle A', and with the second circle B'. Prove that the sum of the two segments AB and $A'B'$ is independent of k, and depends only upon k', i.e. the shift of the circles, and θ.

SOLUTION BY U. G. MITCHELL, University of Kansas.

Let the points A, B, A', B' have coördinates (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), respectively. By the conditions of the problem the points $A = (x_1, y_1)$ and $A' = (x_3, y_3)$ are on the circle $x^2 + y^2 = k^2$, the points $B = (x_2, y_2)$ and $B' = (x_4, y_4)$ are on the circle $(x - k')^2 + y^2 = k^2$, and all four of the points A, B, A', B' are on the line $y = m(x - k')$, where $m = \tan \theta$.

[1] His boasting in the preface to the American edition, in 1876, of *Inventional Geometry: a series of problems*, by his father W. G. Spencer, will be recalled.

Since for a given circle and a given exterior point the product of any secant through the point and the exterior segment of the secant is equal to the square of the tangent from the point to the circle we have $A'B \cdot AB = x_2^2 + y_2^2 - k^2$ and $A'B \cdot A'B' = (x_3 - k')^2 + y_3^2 - k^2$. Hence,

$$AB + A'B' = \frac{x_2^2 + y_2^2 - k^2 + (x_3 - k')^2 + y_3^2 - k^2}{A'B}.$$

By use of $A'B = \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}$ and the relations given above this equation readily reduces to $AB + A'B' = 2k'/\sqrt{1 + m^2}$, which is independent of k and dependent upon k' and θ only, as was to be shown.

Also solved by T. M. Blakslee, Paul Capron, Olive C. Hazlett, C. N. Mills, E. J. Oglesby, Arthur Pelletier, C. P. Sousley, C. E. Stromquist, and H. S. Uhler.

2804 [1920, 32]. Proposed by T. H. GRONWALL, Washington, D. C.

Show that for $|x| < 1$

$$\frac{1}{\sqrt{1 - x^4}} \int_0^x \frac{dx}{\sqrt{1 - x^4}} = x + \sum_1^\infty \frac{3 \cdot 7 \cdots (4n - 5)(4n - 1)}{5 \cdot 9 \cdots (4n - 3)(4n + 1)} x^{4n+1},$$

$$\left(\int_0^x \frac{dx}{\sqrt{1 - x^4}} \right)^2 = x^2 + \sum_1^\infty \frac{3 \cdot 7 \cdots (4n - 5)(4n - 1)}{5 \cdot 9 \cdots (4n - 3)(4n + 1)} \frac{x^{4n+2}}{2n + 1}.$$

Solution by O. S. Adams, Coast and Geodetic Survey.

The general forms for such integrals may be derived and the formulas in the statement will then be special cases of the derived forms. We then state the problem as follows:

For $|x| < 1$, s a positive integer,

$$\frac{1}{\sqrt{1 - x^s}} \int_0^x \frac{dx}{\sqrt{1 - x^s}} = x + \sum_1^\infty \frac{\frac{s+2}{2} \cdot \frac{3s+2}{2} \cdots \frac{(2n-1)s+2}{2}}{(s+1)(2s+1) \cdots (ns+1)} x^{ns+1},$$

$$\left(\int_0^x \frac{dx}{\sqrt{1 - x^s}} \right)^2 = x^2 + \sum_1^\infty \frac{\frac{s+2}{2} \cdot \frac{3s+2}{2} \cdots \frac{(2n-1)s+2}{2}}{(s+1)(2s+1) \cdots (ns+1)} \frac{2x^{ns+2}}{ns+2}.$$

Let

$$\phi(x) = \int_0^x \frac{dx}{\sqrt{1 - x^s}};$$

then

$$\frac{d}{dx} \phi(x) = \frac{1}{\sqrt{1 - x^s}},$$

and

$$2\phi(x) \frac{d}{dx} \phi(x) = \frac{2}{\sqrt{1 - x^s}} \int_0^x \frac{dx}{\sqrt{1 - x^s}}.$$

By integration,

$$\phi^2(x) = 2 \int_0^x \left[\frac{1}{\sqrt{1 - x^s}} \int_0^x \frac{dx}{\sqrt{1 - x^s}} \right] dx.$$

Therefore, if the first series is multiplied by $2dx$ and integrated from 0 to x, the second series will be obtained.

Let

$$y = \frac{1}{\sqrt{1 - x^s}} \int_0^x \frac{dx}{\sqrt{1 - x^s}};$$

then

$$\frac{dy}{dx} = \frac{1}{1 - x^s} + \frac{(s/2)x^{s-1}}{(1 - x^s)^{3/2}} \int_0^x \frac{dx}{\sqrt{1 - x^s}} = \frac{1}{1 - x^s} + \frac{(s/2)x^{s-1}y}{1 - x^s};$$

or

$$(1-x^s)\frac{dy}{dx}-\frac{s}{2}x^{s-1}y-1=0.$$

The function $1/\sqrt{1-x^s}$ can be developed as a power series with unit radius of convergence, this can be integrated term by term, and the result multiplied by the original series, giving for y an expression of the form $y=x+ax^{s+1}+bx^{2s+1}+cx^{3s+1}+dx^{4s+1}+\cdots$. Now the series is absolutely convergent and we can determine the coefficients a, b, c, by substituting this series in the differential equation. We obtain

$$\begin{aligned}1+(s+1)ax^s+(2s+1)bx^{2s}+(3s+1)cx^{3s}+(4s+1)dx^{4s}+\cdots\\ -x^s-(s+1)ax^{2s}-(2s+1)bx^{3s}-(3s+1)cx^{4s}-\cdots\\ -1-\frac{s}{2}x^s-\frac{s}{2}ax^{2s}-\frac{s}{2}bx^{3s}-\frac{s}{2}cx^{4s}-\cdots\equiv 0;\end{aligned}$$

so that the coefficients of the various powers of x must be identically zero. We must then have

$$a=\frac{\frac{s+2}{2}}{s+1},\qquad b=\frac{\frac{s+2}{2}\cdot\frac{3s+2}{2}}{(s+1)(2s+1)},$$

and

$$c=\frac{\frac{s+2}{2}\cdot\frac{3s+2}{2}\cdot\frac{5s+2}{2}}{(s+1)(2s+1)(3s+1)}.$$

The $(n+1)$th coefficient thus becomes equal to the expression

$$\frac{\frac{s+2}{2}\cdot\frac{3s+2}{2}\cdots\frac{(2n-1)s+2}{2}}{(s+1)(2s+1)\cdots(ns+1)}.$$

This proves the formula

$$\frac{1}{\sqrt{1-x^s}}\int_0^x\frac{dx}{\sqrt{1-x^s}}=x+\sum_1^\infty\frac{\frac{s+2}{2}\cdot\frac{3s+2}{2}\cdots\frac{(2n-1)s+2}{2}}{(s+1)(2s+1)\cdots(ns+1)}x^{ns+1}.$$

By multiplying this series by $2dx$ and integrating from 0 to x, we obtain the second formula

$$\left(\int_0^x\frac{dx}{\sqrt{1-x^s}}\right)^2=x^2+\sum_1^\infty\frac{\frac{s+2}{2}\cdot\frac{3s+2}{2}\cdots\frac{(2n-1)s+2}{2}}{(s+1)(2s+1)\cdots(ns+1)}\frac{2x^{ns+2}}{ns+2}.$$

These formulas can easily be verified for $s=1$ and they can be compared with the known forms for $s=2$. When $s=4$, they become identical with those stated in the problem.

$$\frac{1}{\sqrt{1-x^4}}\int_0^x\frac{dx}{\sqrt{1-x^4}}=x+\sum_1^\infty\frac{3\cdot7\cdots(4n-5)(4n-1)}{5\cdot9\cdots(4n-3)(4n+1)}x^{4n+1},$$

$$\left(\int_0^x\frac{dx}{\sqrt{1-x^4}}\right)^2=x^2+\sum_1^\infty\frac{3\cdot7\cdots(4n-5)(4n-1)}{5\cdot9\cdots(4n-3)(4n+1)}\frac{x^{4n+2}}{2n+1}.$$

The last formula is given by Gauss, *Werke*, vol. 3, p. 406.

Also solved by ELIJAH SWIFT and the Proposer.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will cooperate in contributing to the general interest of this department by sending items to the Editor-in-Chief.

Dr. TERESA COHEN, of The Johns Hopkins University, is an instructor of mathematics at Pennsylvania State College.

Dr. MARY G. HASEMAN is an instructor of mathematics at the University of Illinois.

Dr. W. L. CRUM, of Yale University, has been granted leave of absence for the second half of the current academic year to fill a temporary vacancy at Williams College, as assistant professor of mathematics, made by the absence of Professor J. G. HARDY who is on sabbatic leave in Europe.

Assistant Professor P. J. DANIELL, of the Rice Institute, has been promoted to be professor of applied mathematics. During the first semester of the current year he was studying in France.

At Kentucky Wesleyan College, on January 1, 1921, Professor C. S. VENABLE resigned as head of the department of mathematics, and was succeeded by Mr. J. P. BEAMAN, formerly of the Remy Electric Company, Anderson, Ind.

At Baylor University, Waco, Texas, P. C. PORTER, dean of the Baylor College for Women, Belton, Texas, has been appointed assistant professor of mathematics for the coming academic year.

Dr. HARLOW SHAPLEY, of the Mount Wilson Solar Observatory, Pasadena, Cal., an enthusiastic investigator in the structure of the siderial universe, has been appointed observer of the Harvard College Observatory, and he entered upon his duties last March. "The position is a new one; it will give Mr. Shapley an opportunity to take an active part in the work of the Observatory in coöperation with the acting director, Professor S. I. BAILEY, and other members of the staff." (*Harvard Alumni Bulletin*, March 10, 1921.)

The following appointments and promotions were made in the autumn of 1920:

Mr. J. A. HERRINGTON, of the University of Wisconsin, was appointed associate professor of mathematics and engineering at Beloit College.

Mr. R. J. W. TEMPLIN was added, as instructor, to the department of mathematics at Lafayette College, Easton, Pa.

Mr. J. H. TAYLOR, assistant in mathematics at the University of Nebraska, was promoted to an instructorship.

Mr. F. J. ROGERS was appointed head of the department of mathematics at Shaw University, Raleigh, N. C.

Dr. E. D. GRANT, associate professor of mathematics at the Michigan College of Mines, was appointed head of the department of mathematics at Earlham College, Richmond, Ind.

At Carroll College, Waukesha, Wis., Dr. S. B. RAY resigned as head of the department of mathematics and Professor L. S. DANCEY was appointed as his successor.

At Clarkson College of Technology, Potsdam, N. Y., Professor H. M. ROYAL, who was for nine years head of the department of mathematics, resigned, and Mr. T. L. HAMLIN, instructor of mathematics at Union College, Schenectady, N. Y., was appointed as his successor. Mr. L. W. COMSTOCK, Mr. L. A. DE RONDE, Mr. C. V. TERWILLIGER, and Mr. W. L. WARNER, were appointed instructors of mathematics at Union College.

Mr. L. C. BAGLEY, of Washburn College, Topeka, Kan., was appointed professor of mathematics at Ottawa University, Ottawa, Kan., in place of Miss GRACE E. JEFFERSON who was appointed instructor of mathematics at Denison University, Granville, Ohio.

Mr. G. F. ROUSE was appointed instructor of mathematics at Cornell College, Mt. Vernon, Iowa.

At Illinois College, Jacksonville, Ill., Mr. G. H. SCOTT, professor of mathematics and physics, was appointed dean of the college, and Miss LOIS DANIELS, instructor in mathematics and physics.

At the College of Wooster, Wooster, Ohio, Instructor C. O. WILLIAMSON was promoted to be an assistant professor of applied mathematics.

At Alfred College, Alfred, N. Y., A. B. KENYON, professor of mathematics for forty-seven years, retired as dean of the college, and Mr. W. A. TITSWORTH has been transferred from the physics department to become head of the department of mathematics.

At Grove City College, Pa., J. H. BILES resigned as assistant professor of mathematics and physics, to become head of the department of mathematics at Centre College, Danville, Ky. He was succeeded by Mr. G. A. BECKER, formerly with the University of New York City.

At West Virginia Wesleyan College, Professor C. E. WHITE resigned to accept a position at Muskingum College, New Concord, Ohio, and he was succeeded by Instructor THURMAN ANDREW.

At the University of Alabama, Mr. F. A. LEWIS, formerly of the Texas Agricultural and Mechanical College, and Dr. E. KUSCHKE, formerly of the University of California, were appointed assistant professors of mathematics.

At Wesleyan University, Professor FREDERICK SLOCUM, of Brown University, returned as professor of astronomy and director of the Van Vleck Observatory; Mr. C. L. STEARNS, instructor of astronomy and mathematics (*1919*, 86), was appointed to a position in the Yale Observatory.

At Randolph-Macon College, Ashland, Va., Mr. B. F. WALTON was appointed instructor of mathematics.

At The Citadel, the military college of South Carolina, Major R. G. THOMAS, professor of mathematics and engineering for thirty-one years, was made professor emeritus. He was succeeded by Colonel O. J. BOND in the department of mathematics, which is now separate from the department of civil engineering.

At Albion College, Mich., Instructor A. I. BLESS resigned to become instructor of mathematics at the University of Maine, and Professor A. A. MCSWEENEY, of Grand Island College, Neb., was appointed assistant professor in applied mathematics and astronomy.

At Rockford College, Ill., Miss JESSIE M. JACOBS (see *1920*, 337) resigned as associate professor of mathematics and physics, and was succeeded by Instructor STELLA M. ALLEN, of Columbia University.

At Southern College, Clearwater, Fla., W. W. WEBER, professor of mathematics, was appointed dean of the College.

At Goucher College, Baltimore, Md., Associate Professor FLORENCE P. LEWIS was promoted to be professor of mathematics.

At Colgate University, Hamilton, N. Y., Mr. TORVALD FREDERIKSEN was appointed instructor. Assistant Professor E. P. SISSON was granted leave of absence for the year 1920–21.

At Fairmont College, Wichita, Kan., Mr. H. G. TITT, formerly of Yankton College, S. D., was appointed professor of mathematics and dean of the College of Liberal Arts to succeed Professor A. J. HOARE.

At the State College of Washington, Assistant Professor F. N. BRYANT resigned to accept a position in the department of mathematics at Syracuse University; Miss GLADYS H. FREEMAN, of the University of Chicago, and Mr. H. H. IRWIN were appointed instructors of mathematics.

At the College of William and Mary, Williamsburg, Va., Mr. J. S. COUNSELMAN was appointed professor of mathematics and Mr. J. C. LYONS, assistant professor.

At Shorter College, Rome, Ga., Miss JENNIE D. EHLERS, of Columbia University, was appointed instructor of mathematics. Miss RUBY U. HIGHTOWER has been in charge of the department for some time.

At Ripon College, Wis., Mr. WILLIAM BOLLENBECK was appointed instructor of mathematics.

At Villanova College, Pa., Reverend J. S. O'LEARY was appointed professor of mathematics, succeeding Reverend F. A. DRISCOLL, now president of the College. The following have been added to the mathematical staff: Reverend WILLIAM DONOVAN, Mr. E. T. DICKENSON, and Mr. J. A. CRAWFORD.

At Washington and Lee University, Lexington, Va., Mr. EARLE K. PAXTON was appointed assistant professor of mathematics.

Dr. A. H. NORTON, formerly vice-president and head of the department of mathematics in Elmira College, was appointed president and head of the department of mathematics in Keuka College for Women, Keuka Park, N. Y.

At the University of Tennessee, Professor J. B. HAMILTON was made head of the department of mathematics.

At the University of Vermont, Mr. H. G. MILLINGTON and Mr. F. W. HOUSEHOLDER were appointed instructors of mathematics.

At Meredith College, Professor E. F. CANADAY, of the University of South Dakota, was appointed professor of mathematics, succeeding Dr. IDA BARNEY (see *1920*, 336).

At Randolph-Macon Woman's College, Lynchburg, Va., Dean N. A. PATTILLO, head of the department of mathematics, was granted sabbatic leave to study in Paris. Adjunct Professor GILLIE A. LAREW has been acting head of the department. Miss VIRGINIA WATTS, of the University of Chicago, was appointed instructor of mathematics.

At Central Wesleyan College, Warrenton, Mo., Professor J. H. FRICK, head of the department of mathematics for fifty years, retired. He was succeeded by Mr. H. V. KNORR.

At Carleton College, Northfield, Minn., Dr. V. H. WELLS, formerly statistician in the Treasury Department, was appointed assistant professor of mathematics.

At the University of Florida, Mr. P. A. LUCAS was appointed acting assistant professor of mathematics and physics to succeed Mr. WARREN HIGGINS who resigned to become head of the department of mathematics in the Southern Presbyterian College, Clarksville, Tenn.

At Tufts College, Mass., Assistant Professor HARRIS RICE resigned to accept a similar position at Worcester Polytechnic Institute, Mass.; Mr. B. A. HAZELTINE and Miss EDITH L. BUSH were appointed instructors of mathematics.

At the University of Oklahoma, Assistant Professor H. C. GOSSARD resigned to take up work in the Y. M. C. A. and Instructor BETTIE STEPHENS resigned to become a missionary teacher in China. Dr. J. O. HASSLER, of Crane Junior College, Chicago, was appointed associate professor of mathematics and Mr. H. G. LIEBER, Miss ELLA MANSFIELD and Miss DORA MCFARLAND were appointed instructors. Assistant Professor E. D. MEACHAM has been on leave of absence since January, 1921, studying at the University of Chicago; this leave of absence will continue to the end of the next academic year.

JULIUS SCHMITTLE HOFFMAN, instructor of mathematics at Tufts College from September, 1919, died June 29, 1920. Born in New Orleans, La., October 23, 1896, he received the degree Bachelor of Arts from Tulane University, La., in 1917, and the degree Master of Arts from Harvard University in 1918.

MARY WATSON WHITNEY, professor emeritus of astronomy, and from 1889 to 1910 director of the Observatory, of Vassar College, died January 20, 1921. She was born at Waltham, Mass., September 11, 1847. Graduating from Vassar College she studied for a time also at the University of Zürich. In 1881 she became an assistant to Professor Maria Mitchell of Vassar.

WILLIAM THOMPSON SEDGWICK, professor of biology and public health at the Massachusetts Institute of Technology since 1883, died January 25, 1921. He was born at Hartford, Conn., December 29, 1855. In 1917 he collaborated with Professor H. W. TYLER in publishing *A Short History of Science* (New York, Macmillan), which was an "outgrowth of a lecture course given by the authors for several years to undergraduate classes of the Massachusetts Institute of Technology."

SHERBURNE WESLEY BURNHAM, professor of practical astronomy, and astronomer in the Yerkes Observatory, University of Chicago, from 1893 until his re-

tirement in 1914, died March 11, 1921. He was born in Thetford, Vermont, December 12, 1838. He served in several observatories for nearly twenty years before his appointment as astronomer at the Lick Observatory in 1888. From there he went to the Yerkes Observatory. He discovered 1274 new double stars and most of his publications have been in this connection. The 1100-page *General Catalogue of Double Stars within 121° of the North Pole* was published by the Carnegie Institution of Washington in 1906. The same institution published, in 1913, his volume on *Measures of Proper Motion Stars*. His double star observations at the Lick Observatory fill volume 2, 1894, of *Publications of the Lick Observatory of the University of California* and similar observations made at the Dearborn Observatory 1877–1878 occupy nearly 170 pages of volume 44, 1879, of *Memoirs of the Royal Astronomical Society*. More than a score of his minor publications are listed in *Publications of the Members of the University of Chicago 1902–1916*, Chicago, 1917.

ROBERT BELLAMY CLIFTON, professor of experimental philosophy in the University of Oxford 1865–1915, died February 21, 1921 (cf. this MONTHLY *1920*, 274). He was born March 13, 1836, became sixth wrangler in the Cambridge tripos of 1859, and second Smith's prizeman, the senior wrangler and first Smith's prizeman being Canon Wilson (cf. *1920*, 130). His mathematical publications were few and slight. Three may be found in *Quarterly Journal of Mathematics*, 1860; *Transactions of the Cambridge Philosophical Society*, 1866; *Proceedings of the Royal Society of London*, 1877.

MARIE GEORGES HUMBERT, professor at the Ecole Polytechnique and the Collège de France, died January 22, 1921. He was born in Paris, January 7, 1859. Entering the Ecole Normale Supérieure in 1878 he became an agrégé ès sciences mathématiques in 1881 and received the state doctorate from the University of Paris in 1885, his thesis being entitled: *Sur les courbes du genre un*. He succeeded Hermite as a member of the Institute of France in 1901. A survey of his scientific work up to that time was published in pamphlet for ı. He is probably best known to Americans by his two volume *Cour d'analyse professé a l'Ecole Polytechnique*, 1903–1904, where geometric applications are especially emphasized. This work was a development of lithographed sections appearing 1895–1902. His memoirs in *American Journal of Mathematics*, 1888 and 1894, will be recalled. Other biographical details and a portrait may be found in *Acta Mathematica 1882–1912. Table Générale des tomes 1–35*, Upsala, 1913. "Humbert may be compared with Clebsch, because, although he may not have invented a new mathematical engine, he showed unexpected uses of those already provided. In his hands Abel's theorem and Poincaré's researches on Fuchsian functions became magic keys to unlock the treasures of geometry, and give us concrete and elegant images of analytical ideas." ("M." in *Nature*, March 17, 1921).

ACHILLE MARIE GASTON FLOQUET, at different times professor of mathematics, of analysis, and of astronomy, at the University of Nancy since 1880, died October 7, 1920. He was born December 15, 1847. Student at the Ecole

Normale Supérieure 1867–73, agrégé ès sciences mathématiques 1875, and teacher in lycées 1873–78, he was appointed maître de conférence at the University of Nancy in 1878. He received the state doctorate from the University of Paris in 1879, and his elaborate dissertation on the theory of linear differential equations was published in *Annales de l'Ecole Normale Supérieure*, vol. 8, supplement. He developed this subject afterwards in this same periodical, in *Bulletin de la Société des Sciences de Nancy*, and in *Comptes Rendus . . . de l'Académie des Sciences*. It will be recalled that he was the French editor of "Propriétés générales des systèmes d'équations aux dérivées partielles. Equations linèaires du premier ordre" (the original German article being by E. von Weber) in *Encyclopédie des Sciences Mathématiques*, tome 2, vol. 4, fasc. 1, 1913. Thirty-five of his mathematical and astronomical papers, and pamphlets, are listed in *Catalogue générale des livres (imprimés) de la Bibliothèque Nationale*, Paris, vol. 52, 1913. We have already recorded (*1919*, 419) Professor Floquet's retirement from the University of Nancy, as honorary professor. He was also honorary dean of the faculty of sciences.

Professor E. W. BROWN of Yale University was, on January 31, 1921, unanimously elected a corresponding member of the section in astronomy of the Academy of Sciences of the Institute of France (see *1920*, 384). He succeeded the late Professor E. C. Pickering, of Harvard University (*1919*, 134).

For the summer meeting of the Mathematical Association of America the president has appointed the following committees: Program Committee—Professor W. R. LONGLEY (chairman), Professor C. A. FISCHER, Professor E. V. HUNTINGTON, and Professor SARAH E. SMITH; Committee on Arrangements —Professor CLARA E. SMITH (chairman), Professor W. D. CAIRNS, Professor LENNIE P. COPELAND, Professor OLIVE C. HAZLETT, Professor E. J. MILES and Professor H. B. PHILLIPS.

On the occasion of the mathematical congress at Strasbourg last September, the four mathematicians who gave "general lectures" at the congress (*1920*, 439) were elected honorary members of the Society of Sciences, Agriculture, and Arts of the Lower Rhine. It will be recalled that Professor L. E. DICKSON, of the University of Chicago, was one of these lecturers.

Professor H. S. WHITE's three year period as a representative of the American Mathematical Society in the Division of Physical Sciences of the National Research Council expires July 1, and Professor E. B. VANVLECK has been appointed as his successor. The other representatives of the Society are Professors L. E. DICKSON and OSWALD VEBLEN. Professor G. D. BIRKHOFF is a "member at large" of the Division, and Professor E. R. HEDRICK is the representative of the Mathematical Association of America.

As a result of the conference to which we have already made reference (*1921*, 151) there was organized a committee of the National Research Council on the Mathematical Analysis of Statistics. The members of the committee are

Professors J. W. GLOVER, of the University of Michigan, E. V. HUNTINGTON, of Harvard University, T. L. KELLEY, of Leland Stanford University, RAYMOND PEARL, of The Johns Hopkins University, W. M. PERSONS, of Harvard University, and H. L. RIETZ (chairman), of the University of Iowa.

According to the new constitution of the American Association for the Advancement of Science, four instead of five members in addition to the chairman and secretary constitute a sectional committee "which shall in all ways promote the object of the Association within its own field." In Section A this committee is as follows: Professor OSWALD VEBLEN, chairman and a vice-president of the Association, who will preside at Toronto in 1921 and deliver his retiring address in 1922; Professor W. H. ROEVER, secretary until the meeting of 1924–25; Professor DUNHAM JACKSON (January, 1920–December, 1924); Professor A. D. PITCHER (January, 1920–December, 1923); Professor G. A. BLISS (January, 1920–December, 1921); and Professor J. M. PAGE (January, 1920–December, 1921). See also *1921*, 151.

At the Mathematics Section of the New York State Teachers Association which met in Rochester, November 23, 1920, the following papers were read by members of the Association: "Recent tendencies in secondary mathematics" by Mr. FLETCHER DURRELL; "Junior high school mathematics" by Professor J. W. YOUNG; "Motivation of first lessons in junior high school geometry" (illustrated by lantern slides) by Mr. WILLIAM BETZ.

At the annual meeting of the mathematics section of the Washington State Educational Association held at Yakima, October 28, 1920, Professor W. C. EELLS, of Whitman College, spoke on the work of the National Committee on Mathematical Requirements.

At the meeting of the mathematics section of the Nebraska State Teachers Association on November 5, 1920, Professor A. L. CANDY, of the University of Nebraska, delivered an illustrated address on mathematics in European countries, and Professor J. W. YOUNG, of Dartmouth College, discussed the work of the National Committee on Mathematical Requirements. Professor W. C. BRENKE, of the University of Nebraska, was appointed chairman of a committee to distribute throughout Nebraska the reports of the National Committee.

The annual meeting of the mathematics section of the Missouri Society of Science and Mathematics Teachers was held at Kansas City, on November 11, 1920. Mr. ALFRED DAVIS, of the Soldan High School, St. Louis, was elected president, and Miss EULA A. WEEKS, of the Cleveland High School, St. Louis, secretary, for the current year. Professor U. G. MITCHELL, of Kansas University, addressed the meeting on "The unification of elementary mathematics"; and Mr. A. C. ANDREWS, of the Manual Training High School, Kansas City, and Mr. DAVIS spoke regarding the work of the National Committee on Mathematical Requirements and of the National Council of Teachers of Mathematics.

At the mid-winter meeting of the Association of Teachers of Mathematics in New England, in Worcester Academy, Worcester, March 5, 1921, the following

papers were read by members of the Mathematical Association of America: "The teaching of mathematics; its need and place in the curriculum" by Professor F. B. WILLIAMS; "Mathematical principles of educational statistics" by Professor R. W. BURGESS.

At the first meeting of the Connecticut Valley section of the Association of Teachers of Mathematics in New England, in Hartford, November 6, 1920, the following papers were read by members of the Mathematical Association of America: "Preliminary report of the National Committee on College Entrance Requirements" by W. F. DOWNEY; "The mathematics of the freshman year at Yale" by W. R. LONGLEY. The officers for 1921 of the Association of Teachers of Mathematics in New England include the following: President, W. F. DOWNEY, English High School, Boston; vice-president, C. H. CURRIER, Brown University; member of the council, GERTRUDE E. PRESTON, Dana Hall, Wellesley College.

At the fifteenth regular western meeting of the American Mathematical Society, held in Chicago, December 29–30, 1920, the following papers were presented: "Elementary geometry in n dimensions" by R. P. BAKER; "Invariant points under transformations in function space" by G. D. BIRKHOFF and O. D. KELLOGG; "On the complete characterization of the set of points of approximate continuity" by HENRY BLUMBERG; "On the history of symbols for n-factorial" by FLORIAN CAJORI; "Note on the permutability of functions which have the same Schmidt fundamental functions," "Note on convergence in the mean," and "On kernels which have no Fredholm fundamental functions" by E. W. CHITTENDEN; "Applications of algebraic and hypercomplex numbers to the complete solution in integers of quadratic diophantine equations in several variables," "Arithmetic of quaternions," "Determination of all general homogeneous polynomials expressible as determinants with linear elements," and "Homogeneous polynomials with a multiplication theorem" by L. E. DICKSON; "An adaptation of Bing's paradox, involving an arbitrary *a priori* probability" by E. L. DODD; "Boundary value problems with regular singular points (second paper)" by H. J. ETTLINGER; "Fundamental points of potential theory" by G. C. EVANS; "Comparison of different line-geometric representations for functions of a complex variable" by GLADYS E. C. GIBBENS; "Functionals of summable functions" by W. L. HART; "Note on an ambiguous case of approximation," "Note on the convergence of weighted trigonometric series," "On the method of least mth powers for a set of simultaneous equations," and "On the trigonometric representation of an ill-defined function" by DUNHAM JACKSON; A convergence theorem of Osgood's, with an application" by O. D. KELLOGG; "On polynomials and their residue systems (second paper)" by A. J. KEMPNER; "The integrals $\int_0^x e^{x^2/2}dx$, $\int_0^x \sin\frac{x^2}{2}dx$, $\int_0^x \cos\frac{x^2}{2}dx$ and associated divergent series" by W. D. MACMILLAN; "On amicable numbers and their generalizations" by T. E. MASON; "I-conjugate operators of an abelian group" by G. A. MILLER; "A closed connected set of points which contains no simple continuous arc," and "Conditions under which one of two given closed linear point sets may be

thrown into the other one by a continuous transformation of a plane into itself" by R. L. MOORE; "Determination of the spherical transformation in Grassmann's extensive algebra," and "On the relation of iterative compositional equations to Lie's theory of transformation groups" by A. R. SCHWEITZER; "Expansion of the double-frequency function into a series of Hermite's polynomials" by E. R. SMITH; "Construction of doubly periodic functions with singular points in the period parallelogram" by W. P. WEBBER; "Isothermally conjugate nets," and "Transformation of conjugate nets into conjugate nets" by E. J. WILCZYNSKI.

At the United States Naval Academy there are in the department of mathematics, besides two officers in the navy, 3 professors, 7 associate professors, 7 assistant professors, and 21 instructors. Assistant Professor ARTHUR KIERNAN was promoted to an associate professorship last November.

At the University of Washington, the College of Business Administration has made a course in the elements of statistical methods, offered by the department of mathematics, a requirement for all students in the College. This means that twelve sections of the work will have to be offered during the coming year. The required work in mathematics in the College of Business Administration lasts for three quarters, five hours each; two quarters for the theory of investments and one quarter for the course in statistical methods.

Additional announcements (cf. *1921*, 194) of courses in mathematics at Summer Sessions in 1921, are as follows:

The University of Chicago, First Term, June 20–July 27; Second Term, July 28–September 2, 1921. By Professor E. H. MOORE: Hermitian matrices of positive type, 4 hours, and Determinants, 4 hours, first term only. By Professor H. E. SLAUGHT: Definite integrals, 4 hours, and Differential calculus, 5 hours. By Professor L. E. DICKSON: Seminar on algebra and theory of numbers, 2 hours, and Solid analytic geometry, 4 hours. By Professor E. J. WILCZYNSKI: Projective differential geometry, 4 hours, and College algebra, 5 hours. By Professor A. C. LUNN: Applications of vector analysis to electromagnetism, 4 hours, and Units and dimensions, 4 hours. By Professor J. W. A. YOUNG: Selected topics of mathematics, 4 hours, and Integral calculus, 5 hours. By Professor S. LEFSCHETZ: Selected chapters of algebraic geometry, 4 hours, and Plane analytic geometry, 5 hours. By Professor Henry BLUMBERG: Functions of a real variable, 4 hours, and Plane trigonometry, 5 hours.

Cornell University, July 5–August 12. By Professor V. SNYDER: Projective geometry, 5 hours. By Professor W. A. HURWITZ: Advanced analysis, 5 hours. In addition to these courses, provision for thesis work is offered as follows: By Professor SNYDER, in algebraic geometry; by Professor F. R. SHARPE, in Fourier series and the potential theory; by Professors W. B. CARVER and F. W. OWENS, in projective geometry; by Professor C. F. CRAIG, in functions of a complex variable.

University of Illinois, June 20–August 13. In addition to the usual courses in College algebra, Plane trigonometry, Analytic geometry, Differential calculus, and Integral calculus the following advanced courses are offered for graduate and advanced undergraduate students: By Professor A. B. COBLE: Differential geometry, and Projective geometry. By Professor A. J. KEMPNER: Theory of equations and determinants, and Advanced calculus. By Professor G. E. WAHLIN: Introduction to group theory and the Galois theory of equations. Each course meets five hours per week.

University of Iowa, First Term, June 13–July 23. By Professor J. F. REILLY: Teachers' course, and Solid geometry. By Dr. W. H. WILSON: Trigonometry, Calculus, and Theory of equations. By Dr. ROSCOE WOODS: College algebra, Analytic geometry, and Advanced calculus. By Mr. R. E. GLEASON: Solid geometry, and Whittaker and Watson's *Modern Analysis*. Second Term, July 25–August 26. By Professor R. P. BAKER: Analytic geometry, Projective geometry, and Reading course. By Professor CHITTENDEN: Calculus, and Reading course. By Mr. GLEASON: College algebra, and Reading course. The courses are five hours per week in class through the sessions. When transformed into hours of the ordinary academic year the credit for each course taken during the first session is two semester hours and for the second session five-sixths of this amount.

University of Michigan. July 1–August 26. By Professor W. W. BEMAN: Differential equations, and Teachers course in algebra and geometry. By Professor J. W. GLOVER: Finite differences, and Theory of probability. By Professor W. B. FORD: Advanced calculus, and Higher algebra. By Professor P. FIELD: Advanced mechanics. By Professor T. R. RUNNING: Empirical formulas. By Professor L. C. KARPINSKI: History of mathematics. By Professor J. W. BRADSHAW: Calculus, and Advanced analytic geometry. By Professor H. C. CARVER: Introduction to the mathematical theory of interest, and Actuarial exercises and problems. Each course meets four hours per week, credit two hours.

University of Oklahoma, June 8–August 2. By Professor S. W. REAVES: Analytic geometry, 5 hours, and Bôcher's *Higher Algebra*, 3 hours. By Professor J. O. HASSLER: Trigonometry, 3 hours, Differential equations, 3 hours, and Teaching of secondary mathematics, 2 hours. By Professor N. ALTSHILLER-COURT: College algebra, 3 hours, Differential calculus, 3 hours, and Integral calculus, 3 hours.

University of Pennsylvania, July 5–August 13. By Mr. J. M. THOMAS: Solid geometry, and College algebra. By Professor G. H. HALLETT: Plane trigonometry, Integral calculus, and Theory of groups of a finite order. By Professor H. H. MITCHELL: Intermediate algebra, Differential calculus, and Analytic geometry. By Professor F. W. BEAL: Analytic geometry, Advanced plane and spherical trigonometry, and Differential geometry. These courses will be accepted as two thirds of regular semester courses.

THE MARCH MEETING OF THE ROCKY MOUNTAIN SECTION.

The fifth regular meeting of the Rocky Mountain Section was held at the University of Denver, Denver, Colorado, on March 25, 26. Sessions were held on Friday afternoon and Saturday morning. The presiding officer was Professor H. E. RUSSELL of the University of Denver.

The attendance was thirty-three, including the following nineteen members of the Association: E. L. Brown, I. M. DeLong, A. R. Fehn, G. W. Finley, J. C. Fitterer, W. H. Hill, C. A. Hutchinson, H. A. Howe, G. H. Light, F. H. Loud, J. Q. McNatt, S. L. Macdonald, D. H. Menzel, H. E. Russell, C. H. Sisam, C. S. Sperry, C. E. Stromquist, O. B. Trout, J. W. Woodrow.

Those in attendance were royally entertained at a six-thirty dinner and welcomed to the University by Chancellor W. D. Engle. The reply for the Association was made by S. L. Macdonald. At the business meeting which followed, W. H. Hill, of the Greeley High School, was chosen Chairman and G. W. Gorrell, of the Colorado School of Mines, Vice-Chairman for the meeting to be held at the State Teacher's College at Greeley next year. Dean Howe then invited the guests to inspect the observatory, which was greatly enjoyed by all.

The following nine papers were read:

(1) "The effect of polarized light on the photographic plate" by Dr. J. W. Woodrow, professor of physics, University of Colorado.

(2) "The effect of translation upon certain dispersion and correlation formulas" by Professor C. E. Stromquist, University of Wyoming.

(3) "Trajectories" by Mr. Philip Fitch, North Denver High School (by invitation).

(4) "Note on extraneous loci" by Professor G. H. Light, University of Colorado.

(5) "On correspondence between curves" by Professor C. H. Sisam, Colorado College.

(6) "Mathematics of the high school as preparation for college" by Mr. B. F. Kitchen, Colorado Agricultural College (by invitation).

(7) "Thermal properties of glass" by W. B. Pietenpol, associate professor of physics, University of Colorado.

(8) "The suspended chain" by J. Q. McNatt, engineer for the Colorado Fuel and Iron Co.

(9) "A proof of a theorem in the adjustment of observations by the use of determinants" by Professor C. S. Sperry, University of Colorado.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles.

1. It has been suggested by H. S. Allen and others that the action of light on the photographic film is due to a photoelectric effect. If this is true, incident plane polarized light should have different effects in different directions. Professor Woodrow photographed dark lines through a good Nicol's prism and found